Fill Ups, of Vector Algebra & Three Dimensional Geometry

Q. 1. Let Let $\vec{A}$, $\vec{B}$, $\vec{C}$ **be vectors of length 3, 4, 5 respectively. Let** $\vec{A}$ **be perpendicular to** $\vec{B}+\vec{C}, \vec{B}$ to $\vec{C}+\vec{A}$ and $\vec{C}$ *to* $\vec{A}+\vec{B}$. **Then the length of vector** $\vec{A}+\vec{B}+\vec{C}$ is .... **(1981 - 2 Marks)**

Ans. $5\sqrt{2}$

Solution.

Given that $|\vec{A}| = 3$; $|\vec{B}| = 4$; $|\vec{C}| = 5$

$$\vec{A} \perp (\vec{B}+\vec{C}) \Rightarrow \vec{A}.(\vec{B}+\vec{C}) = 0 \Rightarrow \vec{A}.\vec{B}+\vec{A}.\vec{C} = 0 \quad ...(1)$$

$$\vec{B} \perp (\vec{C}+\vec{A}) \Rightarrow \vec{B}.(\vec{C}+\vec{A}) = 0 \Rightarrow \vec{B}.\vec{C}+\vec{B}.\vec{A} = 0 \quad ...(2)$$

$$\vec{C} \perp (\vec{A}+\vec{B}) \Rightarrow \vec{C}.(\vec{A}+\vec{B}) = 0 \Rightarrow \vec{C}.\vec{A}+\vec{C}.\vec{B} = 0 \quad ...(3)$$

Adding (1), (2) and (3) we get

$$2\left(\vec{A}.\vec{B}+\vec{B}.\vec{C}+\vec{C}.\vec{A}\right) = 0 \quad ...(4)$$

Now, $\left|\vec{A}+\vec{B}+\vec{C}\right|^2 = \left(\vec{A}+\vec{B}+\vec{C}\right).\left(\vec{A}+\vec{B}+\vec{C}\right)$

$= \vec{A}.\vec{A}+\vec{B}.\vec{B}+\vec{C}.\vec{C}+2\vec{A}.\vec{B}+2\vec{B}.\vec{C}+2\vec{C}.\vec{A}$

$= |\vec{A}|^2+|\vec{B}|^2+|\vec{C}|^2+2\left(\vec{A}.\vec{B}+\vec{B}.\vec{C}+\vec{C}.\vec{A}\right)$

$= 50$ $\therefore \left|\vec{A}+\vec{B}+\vec{C}\right| = 5\sqrt{2}$

Q. 2. The unit vector perpendicular to the plane determined by P(1, –1, 2), Q (2, 0, –1) and R(0, 2, 1) is .... (1983 - 1 Mark)

Ans. $\pm\frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}$

Solution. Required unit vector, $\hat{n} = \pm\frac{\overrightarrow{PQ}\times\overrightarrow{PR}}{\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right|}$

$$\overrightarrow{PQ} = \hat{i} + \hat{j} - 3\hat{k}\,;\; \overrightarrow{PR} = -\hat{i} + 3\hat{j} - \hat{k}$$

$$\therefore \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -3 \\ -1 & 3 & -1 \end{vmatrix}$$

$$= (-1+9)\hat{i} + (3+1)\hat{j} + (3+1)\hat{k} \;\; = 8\hat{i} + 4\hat{j} + 4\hat{k}$$

$$\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = \sqrt{64+16+16} = \sqrt{96} = 4\sqrt{6}$$

$$\hat{n} = \pm\left(\frac{8\hat{i} + 4\hat{j} + 4\hat{k}}{4\sqrt{6}}\right) = \pm\left(\frac{2\hat{i} + \hat{j} + \hat{k}}{\sqrt{6}}\right)$$

Q. 3. The area of the triangle whose vertices are A (1, –1, 2), B (2, 1, –1), C(3, – 1, 2) is .... (1983 - 1 Mark)

Ans. $\sqrt{13}$

Solution.

$$\text{Area of } \Delta ABC = \frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right|$$

$$\overrightarrow{BA} = -\hat{i} - 2\hat{j} + 3\hat{k}\,,\; \overrightarrow{BC} = \hat{i} - 2\hat{j} + 3\hat{k}$$

$$\therefore \Delta = \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -2 & 3 \\ 1 & -2 & 3 \end{vmatrix} = \frac{1}{2}\left|6\hat{j} + 4\hat{k}\right| = \left|3\hat{j} + 2\hat{k}\right| = \sqrt{9+4} = \sqrt{13}$$

Q. 4. A, B, C and D, are four points in a plane with position vectors a, b, c and d respectively such that

$$(\vec{a} - \vec{d})(\vec{b} - \vec{c}) = (\vec{b} - \vec{d})(\vec{c} - \vec{a}) = 0$$

The point D, then, is the .... of the triangle ABC. (1984 - 2 Marks)

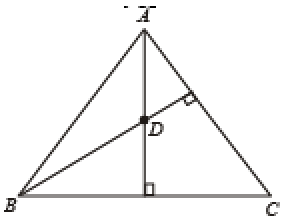
Ans. orthocenter

Solution. Given that $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are position vectors of points A, B, C and D respectively, such that

$\left(\vec{a}-\vec{d}\right).(\vec{b}-\vec{c})=(\vec{b}-\vec{d}).(\vec{c}-\vec{a})=0$

$\Rightarrow \overrightarrow{DA}.\overrightarrow{CB}=\overrightarrow{DB}.\overrightarrow{AC}=0$

$\Rightarrow \overrightarrow{DA}\perp\overrightarrow{CB}$ and $\overrightarrow{DB}\perp\overrightarrow{AC}$



Clearly D is orthocenter of DΔABC

Q. 5. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}=0$ **and the vectors** $\vec{A}=(1,a,a^2)$, $\vec{B}=(1,b,b^2)$, $\vec{C}=(1,c,c^2)$, **are non -coplanar, then the product abc = ...** **(1985 - 2 Marks)**

Ans. –1

Solution. Given that $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix}=0$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}+abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=0$$

Operating $C_2 \leftrightarrow C_3$ and then $C_1 \leftrightarrow C_2$ in first determinant

$$\Rightarrow \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}+abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=0$$

$$\Rightarrow (1+abc)\begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}=0$$

$$\Rightarrow \text{either } 1+abc=0 \text{ or } \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}=0$$

Also given that the vectors $\vec{A},\vec{B},\vec{C}$ are noncoplanar

i.e., $\left[\vec{A}\vec{B}\vec{C}\right]\neq 0$ where $\vec{A}=\hat{i}+a\hat{j}+a^2\hat{k}$

$$\vec{B}=\hat{i}+b\hat{j}+b^2\hat{k},\ \vec{C}=\hat{i}+c\hat{j}+c^2\hat{k}\ \Rightarrow \begin{vmatrix}1 & a & a^2\\ 1 & b & b^2\\ 1 & c & c^2\end{vmatrix}\neq 0$$

∴ We must have 1 + abc = 0 ⇒ abc = – 1

Q. 6. If $\vec{A}\,\vec{B}\,\vec{C}$ are three non-coplanar vectors, then –

$$\frac{\vec{A}\cdot\vec{B}\times\vec{C}}{\vec{C}\times\vec{A}\cdot\vec{B}}+\frac{\vec{B}\cdot\vec{A}\times\vec{C}}{\vec{C}\cdot\vec{A}\times\vec{B}}=\ldots\ldots$$

(1985 - 2 Marks)

Ans. 0

Solution. As given that $\vec{A},\vec{B},\vec{C}$ are three noncoplan ar vectors, therefore, $\left[\vec{A}\vec{B}\vec{C}\right]\neq 0$

Also by the property of scalar triple product we have

$$\vec{A}.(\vec{B}\times\vec{C})=\left[\vec{A}\vec{B}\vec{C}\right],\ \vec{B}.(\vec{A}\times\vec{C})=-\left[\vec{A}\vec{B}\vec{C}\right]$$

$$\vec{C}\times(\vec{A}.\vec{B})=\left[\vec{A}\vec{B}\vec{C}\right],\ \vec{C}.(\vec{A}\times\vec{B})=\left[\vec{A}\vec{B}\vec{C}\right]$$

$$\therefore \frac{\vec{A}.(\vec{B}\times\vec{C})}{(\vec{C}\times\vec{A}).\vec{B}}+\frac{\vec{B}.(\vec{A}\times\vec{C})}{(\vec{C}\times\vec{A}).\vec{B}}=\frac{\left[\vec{A}\,\vec{B}\,\vec{C}\right]}{\left[\vec{A}\,\vec{B}\,\vec{C}\right]}+\frac{-\left[\vec{A}\,\vec{B}\,\vec{C}\right]}{\left[\vec{A}\,\vec{B}\,\vec{C}\right]}=0$$

Q. 7. If $\vec{A} = (1, 1, 1),\ \vec{C} = (0, 1, -1)$ are given vectors, then a vector B satisfying the equations $\vec{A} \times \vec{B} = \vec{C}$ and $\vec{A} \cdot \vec{B} = 3$ ...... **(1985 - 2 Marks)**

Ans. $\frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

Solution.

Given $\vec{A} = \hat{i} + \hat{j} + \hat{k}$, $\vec{C} = \hat{j} - \hat{k}$

Let $\vec{B} = x\hat{i} + y\hat{j} + z\hat{k}$

$$ATQ,\ \vec{A} \times \vec{B} = \vec{C} \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{j} - \hat{k}$$

$$\Rightarrow (z-y)\ \hat{i} + (x-z)\hat{j} + (y-x)\ \hat{k}\ = \hat{j} - \hat{k}$$

$$\Rightarrow \begin{matrix} z-y=0 \\ x-z=1 \\ y-x=-1 \end{matrix} \Rightarrow \left.\begin{matrix} y=z \\ x=1+z \end{matrix}\right\} \quad ...(1)$$

$\vec{A}.\vec{B} = 3 \Rightarrow x + y + z = 3$

Using equations (1) and (2) we get

1 + z + z + z = 3

⇒ z = 2/3 ⇒ y = 2/3, x =5/3

$$\therefore \vec{B} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Q. 8. If the vectors $a\hat{i} + \hat{j} + \hat{k},\ \hat{i} + b\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + c\hat{k}$ **(a ≠ b ≠ c ≠ 1) are coplanar, then the value of** $\frac{1}{(1-a)} + \frac{1}{(1-b)} + \frac{1}{(1-c)} =$ .... **(1987 - 2 Marks)**

Ans. 1

Solution. Given that the vectors $\hat{u}=a\hat{i}+\hat{j}+\hat{k}$, $\hat{v}=\hat{i}+b\hat{j}+\hat{k}$ and $\hat{w}=\hat{i}+\hat{j}+c\hat{k}$ where a ≠ b ≠ c ≠ 1 are coplanar

$$\therefore [\vec{u}\,\vec{v}\,\vec{w}]=0 \Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}=0$$

Oeprating C_1-C_2, C_2-C_3

$$\begin{vmatrix} a-1 & 0 & 1 \\ 1-b & b-1 & 1 \\ 0 & 1-c & c \end{vmatrix}=0$$

Taking (1 – a), (1 – b), (1 – c) common from R_1, R_2 and R_3 respectively.

$$\Rightarrow (1-a)(1-b)(1-c)\begin{vmatrix} -1 & 0 & \frac{1}{1-a} \\ 1 & -1 & \frac{1}{1-b} \\ 0 & 1 & \frac{c}{1-c} \end{vmatrix}=0$$

$$\Rightarrow (1-a)(1-b)(1-c)\left[-\left\{\frac{-c}{1-c}-\frac{1}{1-b}\right\}+\frac{1}{1-a}(1-0)\right]=0$$

$$\Rightarrow (1-a)(1-b)(1-c)\left[\frac{1}{1-a}+\frac{1}{1-b}+\frac{c}{1-c}\right]=0$$

$$\Rightarrow (1-a)(1-b)(1-c)\left[\frac{1}{1-a}+\frac{1}{1-b}-\frac{(1-c)-1}{1-c}\right]=0$$

$$\Rightarrow (1-a)(1-b)(1-c)\left[\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}-1\right]=0$$

But a ≠ b ≠ c ≠ 1 (given)

$$\therefore \frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}-1=0 \Rightarrow \frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=1$$

Q. 9. Let $b=4\hat{i}+3\hat{j}$ and $\vec{c}$ **be two vectors perpendicular to each other in the xy-plane. All vectors in the same plane having projections 1 and 2**

along $\vec{b}$ and $\vec{c}$, respectively,, are given by .... (1987 - 2 Marks)

Ans. $2\hat{i}-\hat{j}$

Solution.

Let $\vec{c}=\alpha\hat{i}+\beta\hat{j}$

As $\hat{b}\perp\hat{c}$ (given) $\therefore\ \vec{b}.\vec{c}=0$

$\Rightarrow(4\hat{i}+3\hat{j}).(\alpha\hat{i}+\beta\hat{j})=0\Rightarrow 4\alpha+3\beta=0$

$\Rightarrow\alpha=-\frac{3\beta}{4}\Rightarrow\frac{\alpha}{+3}=\frac{\beta}{-4}=\lambda$

$\Rightarrow\alpha=+3\lambda,\ \beta=-4\lambda$ ...(1)

Now, let $\vec{a}=x\hat{i}+y\hat{j}$ be the required vectors.

Then as per question

Projection of $\vec{a}$ along $\vec{b}=\frac{\vec{a}.\vec{b}}{|\vec{b}|}\Rightarrow\frac{4x+3y}{\sqrt{4^2+3^2}}=1$

$\Rightarrow$ 4x + 3y = 5 ..(2)

Also, projection of $\vec{a}$ along $\vec{c}=2$

$\Rightarrow\frac{\vec{a}.\vec{c}}{|\vec{c}|}=2\Rightarrow\frac{\alpha x+\beta y}{\sqrt{\alpha^2+\beta^2}}=2\Rightarrow\frac{3\lambda x-4\lambda y}{\sqrt{(3\lambda)^2+(-4\lambda)^2}}=2$

$\Rightarrow 3\lambda x-4\lambda y=10\lambda$

$\Rightarrow$ 3x – 4y = 10 ...(3)

Solving (2) and (3), we get x = 2, y = – 1

$\therefore$ The required vector is $2\hat{i}-\hat{j}$

Q. 10. The components of a vector $\vec{a}$ along and perpendicular to a non-zero

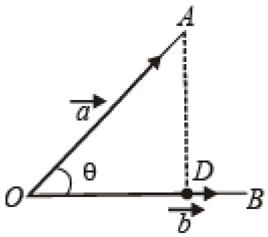
vector $\vec{b}$ are **..........and ....respectively..** **(1988 - 2 Marks)**

Ans. $\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$, $\vec{a}-\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\right)\vec{b}$

Solution. Component of $\vec{a}$ along $\vec{b}$ = $\overline{OD} = OA\cos\theta.\hat{b}$

$$=\left(\frac{\vec{a}.\vec{b}}{|\vec{b}|}\right)\frac{\vec{b}}{|\vec{b}|}=\left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

Component of $\vec{a}$ perpendicular to $\vec{b}$



$$=\overline{DA}=\vec{a}-\overline{OD}\;=\vec{a}-\left(\frac{\vec{a}.\vec{b}}{|\vec{b}|^2}\right)\vec{b}$$

Q. 11. Given

that $\vec{a}$ = (1, 1, 1), $\vec{c}$ = (0, 1, – 1), $\vec{a}.\vec{b}=3$ **and** $\vec{a}\times\vec{b}=\vec{c}$, then $\vec{b}$ = ... **(1991 - 2 Marks)**

Ans. $\frac{5\hat{i}+2\hat{j}+2\hat{k}}{3}$

Solution.

Given $\vec{A}=\hat{i}+\hat{j}+\hat{k}$, $\vec{C}=\hat{j}-\hat{k}$

Let $\vec{B}=x\hat{i}+y\hat{j}+z\hat{k}$

$$ATQ,\;\vec{A}\times\vec{B}=\vec{C}\Rightarrow\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\1&1&1\\x&y&z\end{vmatrix}=\hat{j}-\hat{k}$$

$\Rightarrow (z-y)\,\hat{i} + (x-z)\,\hat{j} + (y-x)\,\hat{k} = \hat{j} - \hat{k}$

$\Rightarrow \begin{matrix} z-y=0 \\ x-z=1 \\ y-x=-1 \end{matrix} \Rightarrow \left.\begin{matrix} y=z \\ x=1+z \end{matrix}\right\}$ ...(1)

$\vec{A}.\vec{B} = 3 \Rightarrow x+y+z=3$

Using equations (1) and (2) we get

1 + z + z + z = 3

⇒ z = 2/3 ⇒ y = 2/3, x =5/3

$\therefore \vec{B} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$

Q. 12. A unit vector coplanar with $\vec{i} + \vec{j} + 2\vec{k}$ and $\vec{i} + 2\vec{j} + \vec{k}$ **and perpendicular to** $\vec{i} + \vec{j} + \vec{k}$ is ..... **(1992 - 2 Marks)**

Ans. $\frac{\hat{j}-\hat{k}}{\sqrt{2}}$ or $\frac{-\hat{j}+\hat{k}}{\sqrt{2}}$

Solution. Let $x\hat{i} + y\hat{j} + z\hat{k}$ be a unit vector, coplanar with $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$ and also perpendicular to $\hat{i} + \hat{j} + \hat{k}$

Then, $\begin{vmatrix} x & y & z \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = 0$

$\Rightarrow -3x + y + z = 0$ .....(i)

and $x + y + z = 0$ .....(ii)

Solving the above by cross multiplication method, we get

$\frac{x}{0} = \frac{y}{4} = \frac{z}{-4}$ or $\frac{x}{0} = \frac{y}{1} = \frac{z}{-1} = \lambda(say)$

$\Rightarrow x = 0, y = \lambda, z = -\lambda$

As $x\hat{i}+y\hat{j}+z\hat{k}$ is a unit vector, therefore

$$0+\lambda^2+\lambda^2=1 \Rightarrow \lambda^2=\frac{1}{2} \Rightarrow \lambda=\pm\frac{1}{\sqrt{2}}$$

∴ The required vector is $\frac{\hat{j}-\hat{k}}{\sqrt{2}}$ or $\frac{-\hat{j}+\hat{k}}{\sqrt{2}}$

Q. 13. A unit vector perpendicular to the plane determined by the points P(1, – 1, 2) Q(2, 0, –1) and R(0, 2, 1) is .... (1994 - 2 Marks)

Ans. $-\left(\frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}\right)$

Solution. We have position vectors of points $P\left(\hat{i}-\hat{j}+2\hat{k}\right)$, $Q\left(2\hat{i}-\hat{k}\right)$,

$R\left(2\hat{j}+\hat{k}\right)$

$$\therefore \overrightarrow{QP}=(\hat{i}-\hat{j}+2\hat{k})-(2\hat{i}-\hat{k})=-\hat{i}-\hat{j}+3\hat{k}$$

$$\therefore \overrightarrow{QR}=2\hat{j}+\hat{k}-2\hat{i}+\hat{k}=-2\hat{i}+2\hat{j}+2\hat{k}$$

Now any vector perpendicular to the plane formed by pts

PQR is given by $\overrightarrow{QP}\times\overrightarrow{QR}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & -1 & 3\\ -2 & 2 & 2\end{vmatrix}=-8\hat{i}-4\hat{j}-4\hat{k}$

∴ Unit vector normal to plane $=\pm\left(\frac{-8\hat{i}-4\hat{j}-4\hat{k}}{\sqrt{64+16+16}}\right)$

$$=\pm\left(\frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{6}}\right)$$

Q. 14. A nonzero vector $\bar{a}$ is parallel to the line of intersection of the plane determined by the vectors $\hat{i}, \hat{i}+\hat{j}$ and the plane determined by the vectors $\hat{i}-\hat{j},\ \hat{i}+\hat{k}$. The angle between $\vec{a}$ and the vector $\hat{i}-2\hat{j}+2\hat{k}$ is .... (1996 - 2 Marks)

Ans. $\frac{\pi}{4}$ or $\frac{3\pi}{4}$

Solution. Eq^n of plane containing vectors $\hat{i}$ and $\hat{i}+\hat{j}$ is

$$[\vec{r}-\hat{i}\ \hat{i}\ \hat{i}+\hat{j}]=0 \Rightarrow \begin{vmatrix} x-1 & y & z \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{vmatrix}=0$$

$\Rightarrow z=0$ ....(1)

Similarly, eq^n of plane containing vectors $\hat{i}-\hat{j}$ and $\hat{i}+\hat{k}$ is

$$[\hat{r}-(\hat{i}-\hat{j})\ \hat{i}-\hat{j}\ \hat{i}+\hat{k}]=0 \Rightarrow \begin{vmatrix} x-1 & y+1 & z \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{vmatrix}=0$$

$\Rightarrow (x-1)(-1-0)-(y+1)(1-0)+z(0+1)=0$

$\Rightarrow -x+1-y-1+z=0$

$\Rightarrow x+y-z=0$ ....(2)

Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

Since $\vec{a}$ is parallel to (1) and (2)

$a_3=0$ and $a_1+a_2-a_3=0 \Rightarrow a_1=-a_2$, $a_3=0$

$\therefore$ a vector in direction of $\vec{a}$ is $\hat{i}-\hat{j}$

Now if θ is the angle between $\vec{a}$ and $\hat{i}-2\hat{j}+2\hat{k}$ then

$$\cos\theta=\pm\frac{1.1+(-1)(-2)}{\sqrt{1+1}\sqrt{1+4+4}}=\pm\frac{3}{\sqrt{2}.3}$$

$$\Rightarrow \cos\theta=\pm\frac{1}{\sqrt{2}} \Rightarrow \theta=\pi/4 \text{ or } 3\pi/4$$

Q. 15. If $\vec{b}$ and $\vec{c}$ are any two non-collinear unit vectors and $\vec{a}$ is any vector,

then $(\vec{a}.\vec{b})\vec{b}+(\vec{a}.\vec{c})\vec{c}+\frac{\vec{a}.(\vec{b}\times\vec{c})}{|\vec{b}\times\vec{c}|}(\vec{b}\times\vec{c})= ....$ **(1996 - 2 Marks)**

Ans. $\vec{a}$

Solution. Let us consider $\vec{b}=\hat{i}$ and $\vec{c}=\hat{j}$ then $\vec{b}\times\vec{c}=\hat{k}$

Let $\vec{a}=x\hat{i}+y\hat{j}+z\hat{k}$

Then, $\left(\vec{a}\vec{b}\right)\vec{b}+(\vec{a}\vec{c})\vec{c}+\frac{\vec{a}\vec{b}\times\vec{c}}{|\vec{b}\times\vec{c}|}(\vec{b}\times\vec{c}) = x\hat{i}+y\hat{j}+z\hat{k} = \hat{a}$

Q. 16. Let OA = a, OB = 10 a + 2b and OC = b where O, A and C are non-collinear points. Let p denote the area of the quadrilateral OABC, and let q denote the area of the parallelogram with OA and OC as adjacent sides. If p = kq, then k = ... (1997 - 2 Marks)

Ans. 6

Solution. q = area of parallelogram with $\overrightarrow{OA}$ and $\overrightarrow{OC}$ as

Adjacent sides $=\left|\overrightarrow{OA}\times\overrightarrow{OC}\right| = \left|\vec{a}\times\vec{b}\right|$

And p = area of quadrilateral OABC

$$=\frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|+\frac{1}{2}\left|\overrightarrow{OB}\times\overrightarrow{OC}\right|$$

$$=\frac{1}{2}\left|\vec{a}\times\left(\overrightarrow{10a}+\overrightarrow{2b}\right)\right|+\frac{1}{2}\left|\left(\overrightarrow{10a}+\overrightarrow{2b}\right)\times\vec{b}\right|$$

$$=\left|\vec{a}\times\vec{b}\right|+5\left|\vec{a}\times\vec{b}\right|=6\left|\vec{a}\times\vec{b}\right| \quad \therefore p = 6q \Rightarrow k = 6$$

True / False of Vector Algebra & Three Dimensional Geometry

True / False

Q. 1. Let $\vec{A}, \vec{B}$ and $\vec{C}$ be unit vectors suppose that $\vec{A}.\vec{B} = \vec{A}.\vec{C} = 0$, and that the angle between $\vec{B}$ and $\vec{C}$ is $\pi/6$. Then $\vec{A} = \pm 2(\vec{B} \times \vec{C})$. (1981 - 2 Marks)

Ans. T

Solution. $\vec{A}, \vec{B}, \vec{C}$ are three unit vectors such that

$\vec{A}.\vec{B} = \vec{A}.\vec{C} = 0$ ...(1)

and angle between $\vec{B}$ and $\vec{C}$ is $\pi/6$.

Now eq. (1) shows that $\vec{A}$ is perpendicular to both $\vec{B}$ and $\vec{C}$.

$\therefore \vec{B} \times \vec{C} \parallel \vec{A} \Rightarrow \vec{B} \times \vec{C} = \lambda \vec{A}$ where λ is any scalar..

$\Rightarrow |\vec{B} \times \vec{C}| = |\lambda \vec{A}| \Rightarrow \sin \pi/6 = \pm \lambda$

(as $\pi/6$ is the angle between $\vec{B}$ & $\vec{C}$)

$\Rightarrow \lambda = \pm \frac{1}{2} \Rightarrow \vec{B} \times \vec{C} = \pm \frac{1}{2} \vec{A} \Rightarrow \vec{A} = \pm 2(\vec{B} \times \vec{C})$

$\therefore$ Given statement is true.

Q. 2. If X. A = 0, X.B = 0, X.C = 0 for some non-zero vector X, then [A B C] = 0 (1983 - 1 Mark)

Ans. T

Solution.

$\vec{X}.\vec{A}=0 \Rightarrow$ either $\vec{A}=0$ or $\vec{X}\perp\vec{A}$
$\vec{X}.\vec{B}=0 \Rightarrow$ either $\vec{B}=0$ or $\vec{X}\perp\vec{B}$
$\vec{X}.\vec{C}=0 \Rightarrow$ either $\vec{C}=0$ or $\vec{X}\perp\vec{C}$

if $\vec{A}$ or $\vec{B}$ or $\vec{C}=0 \Rightarrow [\vec{A}\vec{B}\vec{C}]=0$

Other wise if $\vec{X}\perp\vec{A}, \vec{X}\perp\vec{B}, \vec{X}\perp\vec{C}$ then $\vec{A},\vec{B},\vec{C}$ are coplanar $\Rightarrow [\vec{A}\vec{B}\vec{C}]=0$

∴ Given statement is true.

Q. 3. The points with position vectors a + b, a – b, and a + kb are collinear for all real values of k. (1984 - 1 Mark)

Ans. T

Solution. Let position vectors of pts A, B and C be $\vec{a}+\vec{b}$, $\vec{a}-\vec{b}$ and $\vec{a}+k\vec{b}$ respectively..

Then, $\overrightarrow{AB}=$ p.v. of $B-$ p.v of $A=(\vec{a}-\vec{b})-(\vec{a}+\vec{b})=-2\vec{b}$

Similarly, $\overrightarrow{BC}=\vec{a}+k\vec{b}-\vec{a}+\vec{b}=(k+1)\vec{b}$

Clearly $\overrightarrow{AB}\parallel\overrightarrow{BC}\ \forall k\in R$

⇒ A,B, C are collinear $\forall k\in R$

∴ Statement is true.

Q. 4. For any three vectors $\vec{a}, \vec{b}$, and $\vec{c}$, $(\vec{a}-\vec{b}).(\vec{b}-\vec{c})\times(\vec{c}-\vec{a})=2\vec{a}.\vec{b}\times\vec{c}$. **(1989 - 1 Mark)**

Ans. F

Solution. For any three vectors $\vec{a},\vec{b}$ and $\vec{c}$, we have

$$\text{L.H.S.}=(\vec{a}-\vec{b}).(\vec{b}-\vec{c})\times(\vec{c}-\vec{a})$$
$$=(\vec{a}-\vec{b}).(\vec{b}\times\vec{c}-\vec{b}\times\vec{a}-\vec{c}\times\vec{c}+\vec{c}\times\vec{a})$$
$$=(\vec{a}-\vec{b}).(\vec{b}\times\vec{c}+\vec{a}\times\vec{b}+\vec{c}\times\vec{a})$$

$$=\vec{a}.(\vec{b}\times\vec{c})+\vec{a}.(\vec{a}\times\vec{b})+\vec{a}.(\vec{c}\times\vec{a})\ -\vec{b}.(\vec{b}\times\vec{c})-\vec{b}.(\vec{a}\times\vec{b})-\vec{b}.(\vec{c}\times\vec{a})$$

$= [\vec{a}\,\vec{b}\,\vec{c}] - [\vec{a}\,\vec{b}\,\vec{c}] = 0 \neq \text{R.H.S.}$

∴ The given statement is false.

Subjective Problem of Vector Algebra & 3 D Geometry, (Part -1)

Q. 1. From a point O inside a triangle ABC, perpendiculars OD, OE, OF are drawn to the sides BC, CA, AB respectively. Prove that the perpendiculars from A, B, C to the sides EF, FD, DE are concurrent. (1978)

Solution. Let with respect to O, position vectors of points A, B, C, D, E, F be $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}, \vec{f}$.

Let perpendiculars from A to EF and from B to DF meet each other at H. Let position vector of H be $\vec{r}$. we join CH.

In order to prove the statement given in question, it is sufficient to prove that CH is perpendicular to DE.

Now, as $OD \perp BC \Rightarrow \vec{d}.(\vec{b}-\vec{c})=0$

$\Rightarrow \vec{d}.\vec{b}=\vec{d}.\vec{c}$ ...(1)

As $OE \perp AC \Rightarrow \vec{e}.(\vec{c}-\vec{a})=0 \Rightarrow \vec{e}.\vec{c}=\vec{e}.\vec{a}$ ...(2)

As $OF \perp AB \Rightarrow \vec{f}.(\vec{a}-\vec{b})=0 \Rightarrow \vec{f}.\vec{a}=\vec{f}.\vec{b}$ ...(3)

Also $AH \perp EF \Rightarrow (\vec{r}-\vec{a}).(\vec{e}-\vec{f})=0$

$\Rightarrow \vec{r}.\vec{e}-\vec{r}.\vec{f}-\vec{a}.\vec{e}+\vec{a}.\vec{f}=0$ ...(4)

and $BH \perp FD \Rightarrow (\vec{r}-\vec{b}).(\vec{f}-\vec{d})=0$

$\Rightarrow \vec{r}.\vec{f}-\vec{r}.\vec{d}-\vec{b}.\vec{f}+\vec{b}.\vec{d}=0$ ...(5)

Adding (4) and (5), we get

$\vec{r}.\vec{e}-\vec{a}.\vec{e}+\vec{a}.\vec{f}-\vec{r}.\vec{d}-\vec{b}.\vec{f}+\vec{b}.\vec{d}=0$

$\Rightarrow \vec{r}.(\vec{e}-\vec{d})-\vec{e}.\vec{c}+\vec{d}.\vec{c}=0$

(Using (1), (2) and (3))

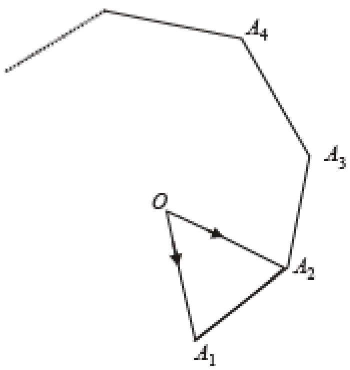
$\Rightarrow \vec{r}.(\vec{e}-\vec{d})-\vec{c}.(\vec{e}-\vec{d})=0 \Rightarrow (\vec{r}-\vec{c}).(\vec{e}-\vec{d})=0$

$\Rightarrow \overrightarrow{CH}.\overrightarrow{ED}=0 \Rightarrow CH \perp ED$ Hence Proved.

Q. 2. A_1, A_2,...................... An are the vertices of a regular plane polygon with n

sides and O is its centre. Show that $\sum_{i=1}^{n-1}(\overrightarrow{OA_i}\times\overrightarrow{OA_{i+1}}) = (1-n)(\overrightarrow{OA_2}\times\overrightarrow{OA_1})$ **(1982 - 2 Marks)**

Solution. $\overrightarrow{OA_1},\overrightarrow{OA_2},...,\overrightarrow{OA_n}$ all vectors are of same magnitude, say 'a' and angle between any two consecutive vector is same that is $\frac{2\pi}{n}$ radians. Let $\hat{p}$ be the unit vectors ⊥ to the plane of the polygon.



$$\therefore \quad \overrightarrow{OA_1}\times\overrightarrow{OA_2} = a^2\sin\frac{2\pi}{n}\hat{p} \qquad ...(i)$$

$$\text{Now, } \sum_{i=1}^{n-1}\overrightarrow{OA_i}\times\overrightarrow{OA_{i+1}} = \sum_{i=1}^{n-1}a^2\sin\frac{2\pi}{n}\hat{p}$$

$$= (n-1)a^2\sin\frac{2\pi}{n}\hat{p} \quad = -(n-1)[\overrightarrow{OA_2}\times\overrightarrow{OA_1}]$$

[using eq^n. (i)]

$$= (1-n)[\overrightarrow{OA_2}\times\overrightarrow{OA_1}] = R.H.S$$

Q. 3. Find all values of λ such that x, y,z, ≠ (0, 0, 0)

and $(\vec{i}+\vec{j}+3\vec{k})x+(3\vec{i}-3\vec{j}+\vec{k})y+(-4\vec{i}+5\vec{j})z = \lambda(x\vec{i}\times\vec{j}\ y+\vec{k}\ z)$ where $\vec{i},\ \vec{j},\ \vec{k}$ **are unit vectors along the coordinate axes. (1982 - 3 Marks)**

Ans. λ = 0, - 1

Solution.

$$(\hat{i}+\hat{j}+3\hat{k})x+(3\hat{i}-3\hat{j}+\hat{k})y+(-4\hat{i}+5\hat{j})z$$
$$=\lambda(x\hat{i}+y\hat{j}+z\hat{k})$$
$$\Rightarrow \quad x+3y-4z=\lambda x \Rightarrow (1-\lambda)x+3y-4z=0$$
$$\Rightarrow \quad x-3y+5z=\lambda y \Rightarrow x-(3+\lambda)y+5z=0$$
$$\Rightarrow \quad 3x+y+0z=\lambda z \Rightarrow 3x+y-\lambda z=0$$

All the above three equations are satisfied for x, y, z not all zero if

$$\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -(3+\lambda) & 5 \\ 3 & 1 & -\lambda \end{vmatrix}=0$$

$$\Rightarrow \quad (1-\lambda)[3\lambda+\lambda^2-5]-3[-\lambda-15]-4[1+9+3\lambda]=0$$
$$\Rightarrow \quad \lambda^3+2\lambda^2+\lambda=0 \Rightarrow \lambda(\lambda+1)^2=0 \Rightarrow \lambda=0,-1.$$

Q. 4. A vector $\vec{A}$ has components A_1, A_2, A_3 in a right -handed rectangular Cartesian coordinate system oxyz. The coordinate system is rotated about the x-axis through an angle $\frac{\pi}{2}$. Find the components of A in the new coordinate system, in terms of A_1, A_2, A_3.

Ans. $A_2\hat{i}-A_1\hat{j}+A_3\hat{k}$

Solution. Since vector $\vec{A}$ has components A_1 , A_2 ,A_3, in the coordinate system OXYZ,

$$\therefore \quad \vec{A}=\hat{i}A_1+\hat{j}A_2+\hat{k}A_3$$

When given system is rotated through $\frac{\pi}{2}$. the new x-axis is along old y-axis and new y-axis is along the old negative x-axis z remains same as before.

Hence the components of A in the new system are

A_2 , - A_1,A_3

$\therefore \quad \vec{A}$ becomes $A_2\hat{i}-A_1\hat{j}+A_3\hat{k}.$

Q. 5. The position vectors of the points A, B, C an d D are $3\hat{i} - 2\hat{j} - \hat{k}, 2\hat{i} + 3\hat{j} - 4\hat{k}, -\hat{i} + \hat{j} + 2\hat{k}$ and $4\hat{i} + 5\hat{j} + \lambda\hat{k},$ **respectively. If the points A, B, C and D lie on a plane, find the value of** λ **.**

Ans. 146/17

Solution. Then $\overrightarrow{AB} = -\hat{i} - 5\hat{j} - 3\hat{k}$, $\overrightarrow{AC} = -4\hat{i} + 3\hat{j} + 3\hat{k}$

$\overrightarrow{AD} = \hat{i} + 7\hat{j} + (1-\lambda)\hat{k}$

We know that A, B, C, D lie in a plane if $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$ are coplanar i.e. $\left[\overrightarrow{AB}\,\overrightarrow{AC}\,\overrightarrow{AD}\right] = 0$

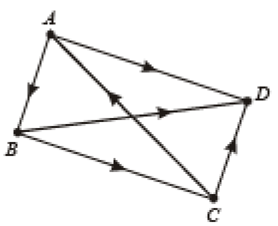
$$\Rightarrow \begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad -1(3 - 3\lambda - 21) - 5(-4 + 4\lambda - 3) - 3(-28 - 3) = 0$$

$$\Rightarrow \quad 3\lambda + 18 - 20\lambda + 35 + 93 = 0 \Rightarrow 17\lambda = 146 \Rightarrow \lambda = \frac{146}{17}$$

Q. 6. If A, B, C , D are any four points in space, prove that – $\left|\overrightarrow{AB} \times \overrightarrow{CD} + \overrightarrow{BC} \times \overrightarrow{AD} + \overrightarrow{CA} \times \overrightarrow{BD}\right| = 4$ **(area of triangle ABC)**

Solution. Let the position vectors of points A, B, C, D be a, b, c, and d respectively with respect to some origin O.



Then, $\overrightarrow{AB} = \vec{b} - \vec{a}$, $\qquad \overrightarrow{AD} = \vec{d} - \vec{a}$,

$\overrightarrow{BC} = \vec{c} - \vec{b}$, $\qquad \overrightarrow{BD} = \vec{d} - \vec{b}$,

$\overrightarrow{CD} = \vec{d} - \vec{c}$, $\qquad \overrightarrow{CA} = \vec{a} - \vec{c}$

Now, $|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$

$=|(\vec{b}-\vec{a})\times(\vec{d}-\vec{c})+(\vec{c}-\vec{b})\times(\vec{d}-\vec{a})+(\vec{a}-\vec{c})\times(\vec{d}-\vec{b})| \quad -\vec{b}\times\vec{d}+\vec{b}\times\vec{a}+\vec{a}\times\vec{d}-\vec{a}\times\vec{b}-\vec{c}\times\vec{d}+\vec{c}\times\vec{b}|$

$=|-\vec{b}\times\vec{c}+\vec{a}\times\vec{c}-\vec{c}\times\vec{a}+\vec{b}\times\vec{a}-\vec{a}\times\vec{b}+\vec{c}\times\vec{b}|$

$=2|\vec{b}\times\vec{a}+\vec{c}\times\vec{b}+\vec{a}\times\vec{c}|$ …(1)

Also Area of ΔABC is

$=\frac{1}{2}|\overrightarrow{BC}\times\overrightarrow{BA}| = \frac{1}{2}|(\vec{c}-\vec{b})\times(\vec{a}-\vec{b})|$

$=\frac{1}{2}|(\vec{c}\times\vec{a}-\vec{c}\times\vec{b}-\vec{b}\times\vec{a}+\vec{b}\times\vec{b}|$

$=\frac{1}{2}|-\vec{b}\times\vec{a}-\vec{c}\times\vec{b}-\vec{a}\times\vec{c}| = \frac{1}{2}|\vec{b}\times\vec{a}+\vec{c}\times\vec{b}+\vec{a}\times\vec{c}|$

$\Rightarrow \quad 2Ar(\Delta ABC)=|\vec{b}\times\vec{a}+\vec{c}\times\vec{b}+\vec{a}\times\vec{c}|$ … (2)

From (1) and (2), we ge

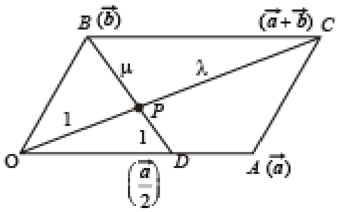
$|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$

$=2(2Ar(\Delta ABC)) \;=4Ar(\Delta\, ABC)$ Hence Proved.

Q. 7. Let OA CB be a parallelogram with O at the origin and OC a diagonal. Let D be the midpoint of OA. Using vector methods prove that BD and CO intersect in the same ratio. Determine this ratio.

Solution. OACB is a parallelogram with O as origin. Let with respect to O position vectors of A and B be $\vec{a}$ and $\vec{b}$ respectively..

Then p.v. of C is $\vec{a}+\vec{b}$.



Also D is mid pt. of OA, therefore position vector of D is $\frac{\vec{a}}{2}$.

CO and BD intersect each other at P.

Let P divides CO in the ratio λ : 1 and BD in the ratio μ : 1 Then by section theorem, position vector of pt. P dividing CO in ratio

$\lambda : 1 = \frac{\lambda \times 0 + 1 \times (\vec{a} + \vec{b})}{\lambda + 1} = \frac{(\vec{a} + \vec{b})}{\lambda + 1}$ ...(1)

And position vector of pt. P dividing BD in the ratio μ : 1 is

$= \frac{\mu\left(\frac{\vec{a}}{2}\right) + 1(\vec{b})}{\mu + 1} = \frac{\mu\vec{a} + 2\vec{b}}{2(\mu + 1)}$ ...(2)

As (1) and (2) represent the position vector of same point, we should have

$$\frac{\vec{a} + \vec{b}}{\lambda + 1} = \frac{\mu\vec{a} + 2\vec{b}}{2(\mu + 1)}$$

Equating the coefficients of $\vec{a}$ and $\vec{b}$, we get

$\frac{1}{\lambda + 1} = \frac{\mu}{2(\mu + 1)}$ ... (i)

$\frac{1}{\lambda + 1} = \frac{1}{\mu + 1}$ ...(ii)

From (ii) we get λ = μ ⇒ P divides CO and BD in the same ratio.

Putting λ = μ in eq. (i) we get μ = 2

Thus required ratio is 2 : 1.

Q. 8. If vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, show that

$$\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \end{vmatrix} = \vec{0}$$

Solution. Given that $\vec{a}, \vec{b}, \vec{c}$ are three coplanar vectors.

∴ There exists scalars x, y, z, not all zero, such that

$x\vec{a} + y\vec{b} + z\vec{c} = \vec{0}$ … (1)

Taking dot product of $\vec{a}$ and (1), we get

$x\vec{a}.\vec{a} + y\vec{a}\vec{b} + z\vec{a}.\vec{c} = \vec{0}$ … (2)

Again taking dot product of $\vec{b}$ and (1), we get

$x\vec{b}.\vec{a} + y\vec{b}.\vec{b} + z\vec{b}.\vec{c} = \vec{0}$ … (3)

Now equations (1), (2), (3) form a homogeneous system of equations, where x, y, z are not all zero.

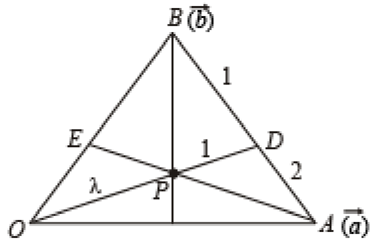
∴ system must have non trivial solution and for this, determinant of coefficient matrix should be zero

i.e. $$\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \end{vmatrix} = 0$$

Hence Proved.

Q. 9. In a triangle OAB, E is the midpoint of BO and D is a point on AB such that AD : DB = 2 : 1. If OD and AE intersect at P, determine the ratio OP : PD using vector methods.

Solution. With O as origin let $\vec{a}$ and $\vec{b}$ be the position vectors of A and B respectively.



Then the position vector of E, the mid point of OB is $\frac{\vec{b}}{2}$.

Again since AD : DB = 2 : 1, the position vector of D is

$$\frac{1.\vec{a}+2\vec{b}}{1+2}=\frac{\vec{a}+2\vec{b}}{3}$$

∴ Equation of OD is

$$\vec{r}=t\left(\frac{\vec{a}+2\vec{b}}{3}\right) \quad \ldots(1)$$

and Equation of AE is

$$\vec{r}=\vec{a}+s\left(\frac{\vec{b}}{2}-\vec{a}\right) \quad \ldots(2)$$

If OD and AE intersect at P, then we will have identical values of $\vec{r}$. Hence comparing the coefficients of $\vec{a}$ and $\vec{b}$, we get

$$\frac{t}{3}=1-s \text{ and } \frac{2t}{3}=\frac{s}{2} \Rightarrow t=\frac{3}{5} \text{ and } s=\frac{4}{5}$$

Putting value of t in eq. (1) we get position vector of point of intersection P as

$$\frac{\vec{a}+2\vec{b}}{5} \quad \ldots(3)$$

Now if P divides OD in the ratio λ : 1, then p.v. of P is

$$\frac{\lambda\left(\frac{\vec{a}+2\vec{b}}{3}\right)+1.0}{\lambda+1}=\frac{\lambda}{3(\lambda+1)}(\vec{a}+2\vec{b}) \quad \ldots(4)$$

From (3) and (4) we get

$$\frac{\lambda}{3(\lambda+1)} = \frac{1}{5} \Rightarrow 5\lambda = 3\lambda + 3 \Rightarrow \lambda = 3/2$$

$\therefore \quad OP : PD = 3:2$

Q. 10. Let $\vec{A} = 2\vec{i} + \vec{k}$, $\vec{B} = \vec{i} + \vec{j} + \vec{k}$, and $\vec{C} = 4\vec{i} - 3\vec{j} + 7\vec{k}$. **Determine a vector** $\vec{R}$. **Satisfying** $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$ and $\vec{R} \cdot \vec{A} = 0$

Ans. $-\hat{i} - 8\hat{j} + 2\hat{k}$

Solution. We are given that $\vec{A} = 2\hat{i} + \hat{k}, \vec{B} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{C} = 4\hat{i} - 3\hat{j} + 7\hat{k}$ and to determine a vector $\vec{R}$ such that $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$ and $\vec{R}.\vec{A} = 0$

Let $\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$

Then $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -3 & 7 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow (y-z)\hat{i} - (x-z)\hat{j} + (x-y)\hat{k} = 10\hat{i} - 11\hat{j} + 7\hat{k}$$

$\Rightarrow$ y -z =-10 … (1)

z -x =-11 … (2)

x -y= 7 … (3)

Also $\vec{R}.\vec{A} = 0$

$\Rightarrow$ 2x +z=0 … (4)

Substituting y =x- 7 and z = -2x from (3) and (4) respectively in eq. (1) we get

x - 7 + 2x = -10 $\Rightarrow$ 3x =-3

$\Rightarrow$ x =-1 , y =-8 and z = 2

$\therefore \quad \vec{R} = -\hat{i} - 8\hat{j} + 2\hat{k}$

Q. 11. Determine the value of 'c' so that for all real x, the vector $cx\hat{i} - 6\hat{j} - 3\hat{k}$ and $x\hat{i} + 2\hat{j} + 2cx\hat{k}$ make an obtuse angle with each other.

Ans. $-\frac{4}{3} < c < 0$

Solution. We have, $\vec{a} = cx\hat{i} - 6\hat{j} + 3\hat{k}$, $\vec{b} = x\hat{i} - 2\hat{j} + 2cx\hat{k}$

Now we know that $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

As angle between $\vec{a}$ and $\vec{b}$ is obtuse, therefore

$\cos\theta < 0 \Rightarrow \vec{a}.\vec{b} < 0$

$\Rightarrow \quad cx^2 - 12 + 6cx < 0 \Rightarrow -cx^2 - 6cx + 12 > 0, \forall x \in R$

$\Rightarrow \quad -c > 0 \text{ and } D < 0 \Rightarrow c < 0 \text{ and } 36c^2 + 48c < 0$

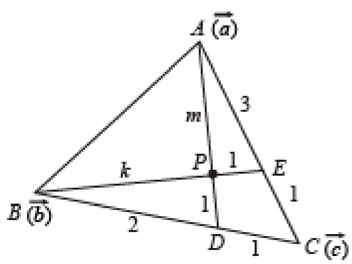
$\Rightarrow \quad c < 0 \text{ and } c(3c+4) < 0 \Rightarrow c < 0 \text{ and } (3c+4) > 0$

$\Rightarrow \quad c < 0 \text{ and } c > -4/3 \Rightarrow -4/3 < c < 0$

Q. 12. In a triangle ABC, D an d E are points on BC and AC respectively, such that BD = 2 DC and AE = 3EC. Let P be the point of intersection of AD and BE. Find BP/PE using vector methods.

Ans. 8 : 3

Solution. Let $\vec{a}, \vec{b}, \vec{c},$ be the position vectors of pt A, B and C respectively with respect to some origin.



ATQ, D divides BC in the ratio 2 : 1 and E divides AC in the ratio 3 : 1.

$\therefore$ position vector of D is $\frac{\vec{b}+2\vec{c}}{3}$ and position vector of E is $\frac{\vec{a}+3\vec{c}}{4}$

Let pt. of intersection P of AD and BE divides BE in the ratio k : 1 and AD in the ratio m : 1, then position vectors of P in these two cases are

$$\frac{\vec{b}+k\left(\frac{\vec{a}+3\vec{c}}{4}\right)}{k+1} \text{ and } \frac{\vec{a}+m\left(\frac{\vec{b}+2\vec{c}}{3}\right)}{m+1} \text{ respectively.}$$

Equating the position vectors of P in two cases we get

$$\frac{k}{4(k+1)}\vec{a}+\frac{1}{k+1}\vec{b}+\frac{3k}{4(k+1)}\vec{c}$$

$$=\frac{1}{m+1}\vec{a}+\frac{m}{3(m+1)}\vec{b}+\frac{2m}{3(m+1)}\vec{c}$$

$$\Rightarrow \quad \frac{k}{4(k+1)}=\frac{1}{m+1} \quad \ldots (1)$$

$$\frac{1}{k+1}=\frac{m}{3(m+1)} \quad \ldots (2)$$

$$\frac{3k}{4(k+1)}=\frac{2m}{3(m+1)}$$

Dividing (3) by (2) we get

$\frac{3k}{4}=2 \Rightarrow k=\frac{8}{3} \Rightarrow$ the req. ratio is 8 : 3.

Q. 13. If the vectors $\vec{b},\vec{c},\vec{d}$, are not coplanar, then prove that the vector $\vec{b},\vec{c},\vec{d}$, $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b})+(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$ is parallel to $\vec{a}$.

Solution. Given that $\vec{b},\vec{c},\vec{d}$ are not coplanar $\therefore [\vec{b},\vec{c},\vec{d}]\neq 0$

Consider, $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})+(\vec{a}\times\vec{c})\times(\vec{d}\times\vec{b}) \quad +(\vec{a}\times\vec{d})\times(\vec{b}\times\vec{c})$

Here, $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})=-(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b})$

$= -(\vec{c} \times \vec{d}.\vec{b})\vec{a} + (\vec{c} \times \vec{d}.\vec{a})\vec{b}$

$= [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$ ...(1)

$(\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) = -(\vec{d} \times \vec{b}) \times (\vec{a} \times \vec{c})$

$= -(\vec{d} \times \vec{b}.\vec{c})\vec{a} + (\vec{d} \times \vec{b}.\vec{a})\vec{c}$

$= [\vec{a}\,\vec{d}\,\vec{b}]\vec{c} - [\vec{c}\,\vec{d}\,\vec{b}]\vec{a}$ ...(2)

$(\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c}) = \left((\vec{a} \times \vec{d})\vec{c}\right)\vec{b} - (\vec{a} \times \vec{d}.\vec{b})\vec{c}$

$= -[\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{a}\,\vec{d}\,\vec{b}]\vec{c}$ ...(3)

[NOTE : Here we have tried to write the given expression in such a way that we can get terms involving $\vec{a}$ and other terms similar which can get cancelled.]

Adding (1), (2) and (3), we get given vector $= -2[\vec{b}\,\vec{c}\,\vec{d}]\vec{a} = k\vec{a}$

⇒ given vector = some constant multiple of $\vec{a}$

⇒ given vector is parallel to $\vec{a}$.

Q. 14. The position vectors of the vertices A, B an d C of a tetrahedron ABCD are $\hat{i} + \hat{j} + \hat{k}, \hat{i}$ and $3\hat{i}$, respectively. The altitude from vertex D to the opposite face ABC meets the median line through A of the triangle ABC at a point E. If the length of the side AD is 4 and the volume of the tetrahedron is $\frac{2\sqrt{2}}{3}$, find the position vector of the point E for all its possible positions.

Ans. (-1, 3, 3) or (3, - 1, - 1)

Solution. We are given AD = 4

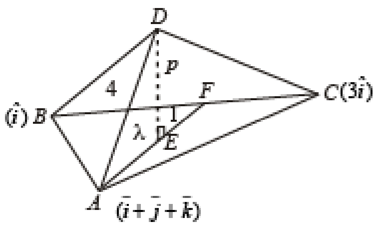
Volume of tetrahedron $= \frac{2\sqrt{2}}{3}$

$$\Rightarrow \quad \frac{1}{3} Ar(\Delta ABC) p = \frac{2\sqrt{2}}{3}$$

$$\therefore \quad \frac{1}{2} |\overrightarrow{BA} \times \overrightarrow{BC}| \, p = 2\sqrt{2}$$

$$\frac{1}{2} |(\hat{j}+\hat{k}) \times 2i| \, p = 2\sqrt{2} \text{ or } |\hat{j}-\hat{k}| \, p = 2\sqrt{2}$$

$$\text{or} \quad \sqrt{2} p = 2\sqrt{2} \quad \therefore p = 2$$



We have to find the P.V. of point E. Let it divides median AF in the ratio λ : 1

$$\therefore \quad \text{P. V. of } E \text{ is } \frac{\lambda.2\hat{i} + (\hat{i}+\hat{j}+\hat{k})}{\lambda+1} \qquad \ldots(2)$$

$$\therefore \quad \overrightarrow{AE} = \text{P.V. of } E - \text{P.V. of } A = \frac{\lambda}{\lambda+1}(\hat{i}-\hat{j}-\hat{k})$$

$$\therefore \quad |\overrightarrow{AE}|^2 = \overrightarrow{AE}^2 = \left(\frac{\lambda}{\lambda+1}\right)^2 .3 \qquad \ldots(3)$$

$$\text{Now, } p^2 + AE^2 = AD^2$$

$$\text{or} \quad 4 + \left(\frac{\lambda}{\lambda+1}\right)^2 .3 = 16 \qquad \therefore 3\left(\frac{\lambda}{\lambda+1}\right)^2 = 12$$

$$\text{or} \quad \left(\frac{\lambda}{\lambda+1}\right) = \pm 2$$

$$\lambda = \pm(2\lambda+2) \qquad \therefore \lambda = -2 \text{ or } -2/3$$

Q. 15. If A, B and C are vectors such that | B | = | C |. Prove that [(A + B) × (A + C)] × (B × C) (B + C) = 0 .

Solution. We have, $(\vec{A}+\vec{B}) \times (\vec{A}+\vec{C})$

$= \vec{A}\times\vec{A}+\vec{B}\times\vec{A}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C}$

$= \vec{B}\times\vec{A}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C}$ $\quad [\because \vec{A}\times\vec{A}=0]$

Thus, $[(\vec{A}+\vec{B})\times(\vec{A}+\vec{C})]\times(\vec{B}\times\vec{C})$

$= [\vec{B}\times\vec{A}+\vec{A}\times\vec{C}+\vec{B}\times\vec{C}]\times(\vec{B}\times\vec{C})$

$= (\vec{B}\times\vec{A})\times(\vec{B}\times\vec{C})+(\vec{A}\times\vec{C})\times(\vec{B}\times\vec{C})$ $\quad [\because x\times x=0]$

$= \left\{(\vec{B}\times\vec{A}).\vec{C}\right\}\vec{B}-\left\{(\vec{B}\times\vec{A}).\vec{B}\right\}\vec{C} \quad +\left\{(\vec{A}\times\vec{C}).\vec{C}\right\}\vec{B}-\left\{(\vec{A}\times\vec{C}).\vec{B}\right\}\vec{C}$

[∵ (a x b) x c = (a.c)b- (b.c)a]

$= [\vec{B}\,\vec{A}\,\vec{C}]\vec{B}-[\vec{A}\,\vec{C}\,\vec{B}]C$

[∵ [A B C] = 0 if any two of A, B, C are equal.]

$= [\vec{A}\,\vec{C}\,\vec{B}]\left\{\vec{B}-\vec{C}\right\}$

Thus, LHS of the given expression

$= [\vec{A}\,\vec{C}\,\vec{B}]\{(\vec{B}-\vec{C}).(\vec{B}+\vec{C})\}$

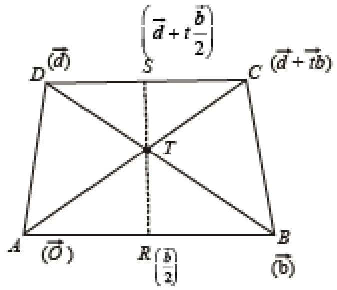
$= [\vec{A}\,\vec{C}\,\vec{B}]\{|\vec{B}|^2-|\vec{C}|^2\}=0$ $\quad$ [∵| B | = C|]

Subjective Problem of Vector Algebra & 3 D Geometry, (Part -2)

Q. 16. Prove, by vector methods or otherwise, that the point of intersection of the diagonals of a trapezium lies on the line passing through the mid-points of the parallel sides. (You may assume that the trapezium is not a parallelogram.) (1998 - 8 Marks)

Solution. The P.Vs. of the points A, B, C, D are

$A(\vec{O}), B(\vec{b}), D(\vec{d}), C(\vec{d}+t\vec{b})$



Equations of AC and BD are

$r = \lambda(d + tb)$ and $r = (1-\mu)b + \mu d$

For point of intersection say T compare the coefficients

$\lambda = \mu,\ t\lambda = 1-\mu = 1-\lambda$ or $(t+1)\lambda = 1$

$\therefore\quad \lambda = \frac{1}{t+1} = \mu$

$\therefore\quad T$ is $\frac{d+tb}{t+1}$ ...(1)

Let R and S be mid-points of parallel sides AB and DC then

R is $\frac{b}{2}$ and S is $d + t\frac{b}{2}$.

Equation of RS by r = a + s(b-a) is

$$r = \frac{b}{2} + s\left[d + (t-1)\frac{b}{2}\right]$$

The point (1) will lie on above if,

$$\frac{d + tb}{1+t} = \frac{b}{2} + s\left[d + (t-1)\frac{b}{2}\right]$$

Comparing the coefficients, we get

$$\frac{t}{1+t} = \frac{1}{2} + s\frac{(t-1)}{2} \text{ and } \frac{t}{(1+t)} = s,$$

$$\therefore \quad \frac{t}{1+t} = \frac{1}{2} + \frac{1}{1+t}.\frac{(t-1)}{2} = \frac{2t}{2(1+t)} = \frac{t}{1+t}$$

Which is true . Hence proved.

Q. 17. For any two vectors u and v, prove that (1998 - 8 Marks)

(a) $(u \cdot v)^2 + |u \times v|^2 = |u|^2 |v|^2$ and

(b) $(1 + |u|^2)(1 + |v|^2) = (1 - u \cdot v)^2 + |u + v + (u \times v)|^2$.

Solution. We have, $\vec{u}.\vec{v} = |\vec{u}||\vec{v}|\cos\theta$ and $\vec{u} \times \vec{v} = |\vec{u}||\vec{v}|\sin\theta\,\hat{n}$ Where θ is the angle between $\vec{u}$ and $\vec{v}$ and $\hat{n}$ is a unit vector perpendicular to both $\vec{u}, \vec{v}$ and is such that $\vec{u}, \vec{v}, \hat{n}$ form a right handed system.

Thus, $|\vec{u}.\vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2 \cos^2\theta$

and $|\vec{u} \times \vec{v}|^2 = |\vec{u}|^2 |\vec{v}|^2 \sin^2\theta\,\hat{n}.\hat{n} = |\vec{u}|^2 |\vec{v}|^2 \sin^2\theta$

$\therefore \quad |u \,.\, v|^2 + |u \times v|^2 = |u|^2 |v|^2 (\cos^2\theta + \sin^2\theta)$
$= |u|^2 |v|^2$

(b) Let $|u| = a, |v| = b, u \times v = ab \sin\theta\,\hat{n}$, where $\hat{n}$ is perpendicular to both u and v , $|a|^2 = a^2$
L.H.S. = $(1 + a^2)(1 + b^2)$

R.H.S. = $(1 - ab\cos q\theta)^2 + (u + v)^2 (u \times v)^2$ $+2(u+v).ab\sin\theta\,\hat{n}$

$= 1 + a^2b^2\cos^2\theta - 2ab\cos\theta + a^2 + b^2 + 2ab\cos\theta + a^2b^2\sin^2\theta.1 + 0$

as $\hat{n}$ is $\perp$ to both u and v.

$= 1 + a^2b^2(\cos^2\theta + \sin^2\theta) + a^2 + b^2$

$= 1 + a^2 + b^2 + a^2b^2 = (1+a^2)(1+b^2)$

Q. 18. Let u an d v be unit vector s. If w is a vector such that w +(w × u) = v, then prove that |(u × v) · w| ≤ 1/2 and that the equality holds if and only if u is perpendicular to v. (1999 - 10 Marks)

Solution. $[\vec{u}\,\vec{v}\,\vec{w}] = (\vec{u}\times\vec{v}).(\vec{v}-\vec{w}\times\vec{u}) = (\vec{u}\times\vec{v}).(\vec{u}\times\vec{w})$

$= \begin{vmatrix} \vec{u}.\vec{u} & \vec{u}.\vec{w} \\ \vec{v}.\vec{u} & \vec{v}.\vec{w} \end{vmatrix}$

Now, $\vec{u}.\vec{u} = 1$

$\vec{u}.\vec{w} = \vec{u}.(\vec{v}-\vec{w}\times\vec{u}) = \vec{u}.\vec{v} - [\vec{u}\,\vec{w}\,\vec{u}] = \vec{u}.\vec{v}$

$\vec{v}.\vec{w} = \vec{v}.(\vec{v}-\vec{w}\times\vec{u}) = 1 - [\vec{v}\,\vec{w}\,\vec{u}] = 1 - [\vec{u}\,\vec{v}\,\vec{w}]$

$\therefore \quad [\vec{u}\,\vec{v}\,\vec{w}] = \begin{vmatrix} 1 & \cos\theta \\ \cos\theta & 1-(\vec{u}\,\vec{v}\,\vec{w}) \end{vmatrix},$

(θ is angle between $\vec{u}$ and $\vec{v}$)

$= 1 - [\vec{u}\,\vec{v}\,\vec{w}] - \cos^2\theta$

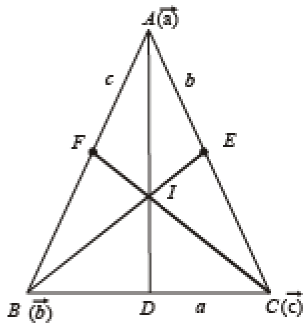
$\therefore \quad [\vec{u}\,\vec{v}\,\vec{w}] = \frac{1}{2}\sin^2\theta \le \frac{1}{2}$

Equality holds when $\sin^2\theta = 1$ i.e., $\theta = \pi/2 \therefore \vec{u} \perp \vec{v}.$

Q. 19. Show, by vector methods, that the angular bisectors of a triangle are concurrent and find an expression for the position vector of the point of concurrency in terms of the position vectors of the vertices. (2001 - 5 Marks)

Solution. Let $\vec{a}, \vec{b}, \vec{c}$ be the position vectors by A, B, and C respectively,

Let AB, BE and CF be the bisectors of ∠A, ∠B , and ∠C respectively.



a, b, c are the lengths of sides BC, CA and AB respectively.

Now we know by angle bisector thm that AD divides, BC in the ratio

BD : DC = AB : AC = c : b.

∴ The position vector of D is $\vec{d} = \dfrac{b\vec{b} + c\vec{c}}{b+c}$

Let I be the point of intersection of BE and AD. Then in ΔABD, BI is bisector of ∠B.

∴ DI : IA = BD : BA

$$\text{But } \frac{BD}{DC} = \frac{c}{b} \Rightarrow \frac{BD}{BD+DC} = \frac{c}{c+b}$$

$$\Rightarrow \frac{BD}{BC} = \frac{c}{c+b} \Rightarrow BD = \frac{ac}{b+c}$$

$$\therefore DI : IA = \frac{ac}{b+c} : c = a : (b+c)$$

$$\therefore \text{P.V. of } I = \frac{\vec{a}.\,a + \vec{d}(b+c)}{a+b+c}$$

$$= \frac{a\vec{a} + \left(\dfrac{b\vec{b}+c\vec{c}}{b+c}\right)(b+c)}{a+b+c} = \frac{a\vec{a}+b\vec{b}+c\vec{c}}{a+b+c}$$

As p.v. of I is symm. in $\vec{a}, \vec{b}, \vec{c}$ and a,b,c.

∴ It must lie on CF as well.

[We can also see that p.v. of intersection of AD and CF is also $\dfrac{a\vec{a}+b\vec{b}+c\vec{c}}{a+b+c}$]

Above prove that all the ∠ bisectors pass through I, i.e., these are concurrent.

Q. 20. Find 3-dimensional vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ satisfying $\vec{v}_1.\vec{v}_1 = 4, \vec{v}_1.\vec{v}_2 = -2, \vec{v}_1.\vec{v}_3 = 6, \vec{v}_2.\vec{v}_2$

$= 2, \vec{v}_2.\vec{v}_3 = -5, \vec{v}_3.\vec{v}_3 = 29$ **(2001 - 5 Marks)**

Ans. $\vec{v_1} = 2\hat{i}\ ; \vec{v_2} = -\hat{i} \pm \hat{j}\ ; \vec{v_3} = 3\hat{i} \pm 2\hat{j} \pm 4\hat{k}$ are some possible values.

Solution. Given data is insufficient to uniquely determine the three vectors as there are only 6 equations involving 9 variables.

∴ We can obtain infinitely many set of three vectors,

$\vec{v}_1, \vec{v}_2, \vec{v}_3$, satisfying these conditions.

From the given data, we get

$$\vec{v}_1 . \vec{v}_1 = 4 \Rightarrow |\vec{v}_1| = 2$$
$$\vec{v}_2 . \vec{v}_2 = 2 \Rightarrow |\vec{v}_2| = \sqrt{2}$$
$$\vec{v}_3 . \vec{v}_3 = 29 \Rightarrow |\vec{v}_3| = \sqrt{29}$$

Also $\vec{v}_1 . \vec{v}_2 = -2 \Rightarrow |\vec{v}_1|\ |\vec{v}_2| \cos\theta = -2$

[Where θ is the angle between $\vec{v}_1$ and $\vec{v}_2$]

$$\Rightarrow \cos\theta = \frac{-1}{\sqrt{2}} \Rightarrow \theta = 135°$$

Now since any two vectors are always coplanar, let us suppose that $\vec{v}_1$ and $\vec{v}_2$ are in x-y plane.

Let $\vec{v}_1$ is along the positive direction of x-axis

then $\vec{v}_1 = 2\hat{i}.$ $[\because |\vec{v}_1| = 2]$

As $\vec{v}_2$ makes an angle 135° with $\vec{v}_1$ and lines in x-y plane, Also keeping in mind

Again let $\vec{v}_3 = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$

$\because \quad \vec{v}_3 . \vec{v}_1 = 6 \Rightarrow 2\alpha = 6 \Rightarrow \alpha = 3$

and $\vec{v}_3 . \vec{v}_2 = -5 \Rightarrow -\alpha \pm \beta = -5 \Rightarrow \beta = \pm 2$

Also $|\vec{v}_3| = \sqrt{29} \Rightarrow \alpha^2 + \beta^2 + \gamma^2 = 29 \Rightarrow \gamma = \pm 4$

Hence $\vec{v}_3 = 3\hat{i} \pm 2\hat{j} \pm 4\hat{k}$

Thus, $\vec{v}_1 = 2\hat{i}\,;\ \vec{v}_2 = -\hat{i} \pm \hat{j}\,; \vec{v}_3 = 3\hat{i} \pm 2\hat{j} \pm 4\hat{k}$

Are some possible answers.

Q. 21. Let $\vec{A}(t) = f_1(t)\hat{i} + f_2(t)\hat{j}$ and $\vec{B}(t) = g_1(t)\hat{i} + g_2(t)\hat{j}, t \in [0,1]$, **where f_1, f_2, g_1, g_2 are continuous functions. If** $\vec{A}(t)$ **and** $\vec{B}(t)$ **are nonzero vectors for all t and** $\vec{A}(0) = 2\hat{i} + 3\hat{j}$, $\vec{A}(1) = 6\hat{i} + 2\hat{j}$, $\vec{B}(0) = 3\hat{i} + 2\hat{j}$ and $\vec{B}(1) = 2\hat{i} + 6\hat{j}$. **Then show that** $\vec{A}(t)$ and $\vec{B}(t)$ **are parallel for some t. (2001 - 5 Marks)**

Solution. $\vec{A}(t)$ is parallel to $\vec{B}(t)$ for some $t \in [0,1]$ if and only if

$$\frac{f_1(t)}{g_1(t)} = \frac{f_2(t)}{g_2(t)} \text{ for some } t \in [0,1]$$

or $f_1(t).g_2(t) = f_2(t).g_1(t)$ for some $t \in [0,1]$

Let $h(t) = f_1(t).g_2(t) - f_2(t).g_1(t)$

$h(0) = f_1(0).g_2(0) - f_2(0).g_1(0)$

$= 2 \times 2 - 3 \times 3 = -5 < 0$

$h(1) = f_1(1).g_2(1) - f_2(1).g_1(1)$

$= 6 \times 6 - 2 \times 2 = 32 > 0$

Since h is a continuous function, and $h(0).h(1) < 0$

$\Rightarrow$ There is some $t \in [0,1]$ for which $h(t) = 0$ i.e., $\vec{A}(t)$ and $\vec{B}(t)$ are parallel vectors for this t.

Q. 22. Let V be the volume of the parallelepiped formed by

the vectors $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. If a_r, b_r, c_r where r = 1, 2, 3, are n on **negative real numbers and** $\sum_{r=1}^{3}(a_r + b_r + c_r) = 3L$, **show that** $V \le L^3$. **(2002 - 5 Marks)**

Solution.

$\therefore$ We have $V = [\vec{a}\ \vec{b}\ \vec{c}\,] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$\Rightarrow$ V = $(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1)$ ....(1)

Now we know that AM $\geq$ GM

$$\therefore \frac{(a_1+b_1+c_1)+(a_2+b_2+c_2)+(a_3+b_3+c_3)}{3}$$

$$\geq [(a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)]^{1/3}$$
$$\Rightarrow \frac{3L}{3} \geq [(a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)]^{1/3}$$
$$\Rightarrow L^3 \geq (a_1+b_1+c_1)(a_2+b_2+c_2)(a_3+b_3+c_3)$$
$$\Rightarrow L^3 \geq a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 + 24 \text{ more such terms}$$

$$L^3 \geq a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 \quad [\because a_r, b_r, c_r \geq 0 \text{ for } r = 1, 2, 3]$$
$$L^3 \geq (a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2) - (a_1b_3c_2 + a_2b_1c_3 + a_3b_2c_1)$$

[Same reason]

$L^3 > V$ from (1) Hence Proved.

Q. 23. (i) Find the equation of the plane passing through the points (2, 1, 0), (5, 0, 1) and (4, 1, 1). (2003 - 4 Marks)
(ii) If P is the point (2, 1, 6) then find the point Q such that PQ is perpendicular to the plane in (i) and the midpoint of PQ lies on it.

Ans. (i) x + y - 2z = 3

(ii) Q(6, 5, - 2)

Solution. (i) Plane passing through (2, 1, 0) , (5, 0, 1) and (4, 1, 1) is

$$\begin{vmatrix} x-2 & y-1 & z-0 \\ 5-2 & 0-1 & 1-0 \\ 4-2 & 1-1 & 1-0 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-2 & y-1 & z \\ 3 & -1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = 0$$

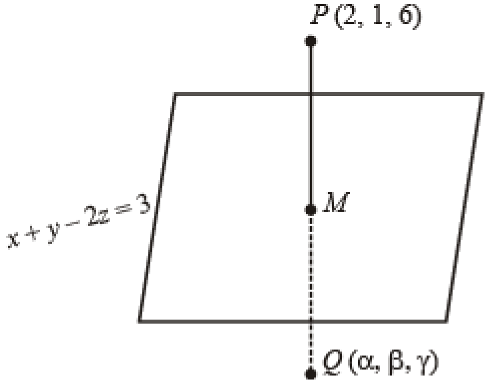
⇒ (x - 2)(-1 - 0) - (y - 1)(3- 2) + z(0 - (-2)) = 0

⇒ -x + 2 - y +1+ 2z = 0 ⇒ x + y - 2z = 3

(ii) As per question we have to find a pt. Q such that PQ is ⊥ to the plane x + y - 2z = 3 ...(1)

And mid pt. of PQ lies on the plane, (Clearly we have to find image of pt. P with respect to plane).

Let Q be (α, β,γ)



Eq^n of PM passing through P(2, 1, 6) and ⊥ to plane x + y - 2z = 3 , is given by

$$\frac{x-2}{1} = \frac{y-1}{1} = \frac{z-6}{-2} = \lambda$$

For some value of λ, $Q(\alpha, \beta, \gamma)$ lies on PM

$\therefore \quad \frac{\alpha-2}{1}=\frac{\beta-1}{1}=\frac{\gamma-6}{-2}=\lambda$

$\Rightarrow \quad \alpha=\lambda+2,\ \beta=\lambda+1,\ \gamma=-2\lambda+6$

$\therefore$ Mid. pt. of PQ

i.e. $M\left(\frac{2+\lambda+2}{2},\frac{1+\lambda+1}{2},\frac{6-2\lambda+6}{2}\right)$

$=\left(\frac{\lambda+4}{2},\frac{\lambda+2}{2},\frac{12-2\lambda}{2}\right)$

But M lies on plane (1)

$\therefore \quad \frac{\lambda+4}{2}+\frac{\lambda+2}{2}-12-2\lambda=3$

$\Rightarrow \lambda+4+\lambda+2-24+4\lambda=6 \Rightarrow 6\lambda=24 \quad \Rightarrow \lambda=4$

$\therefore \quad Q(4+2,4+1,-8+6)=(6,5,-2)$

Q. 24. If $\vec{u}, \vec{v}, \vec{w},$ are three non-coplanar unit vectors and $\alpha,\ \beta,\ \gamma$ are the angles between $\vec{u}$ and $\vec{v}$ and $\vec{w}$, $\vec{w}$ and $\vec{u}$ respectively and $\vec{x}, \vec{y}, \vec{z}$ are unit vectors along the bisectors of the angles $\alpha,\ \beta,\ \gamma$ respectively. Prove

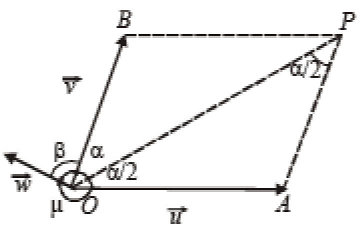
that $[\vec{x}\times\vec{y}\ \ \vec{y}\times\vec{z}\ \ \vec{z}\times\vec{x}]=\frac{1}{16}[\vec{u}\ \vec{v}\ \vec{w}]^2\sec^2\frac{\alpha}{2}\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$. **(2003 - 4 Marks)**

Solution. Given that $\vec{u},\vec{v},\vec{\omega}$ are three non coplanar unit vectors.

Angle $\vec{u}$ and $\vec{v}$ is α, between $\vec{v}$ and $\vec{\omega}$ is β and

between $\vec{\omega}$ and $\vec{u}$ it is γ. In fig. $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent $\vec{u}$ and $\vec{v}$.

Let P be a pt. on angle bisector of ∠AOB such that OAPB is a parallelogram



Also ∠POA =∠BOP = α /2

$\therefore \angle APO = \angle BOP = \alpha/2$ (alt. int. $\angle$'s)

$\therefore$ In ΔOAP, OA= AP a unit vector in the direction of $\overrightarrow{OP}$

$\therefore \quad \overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP} = \vec{u} + \vec{v}$

$\therefore$A unit vector in the direction of

$\overrightarrow{OP} = \dfrac{\vec{u}+\vec{v}}{|\vec{u}+\vec{v}|} \qquad$ i.e. $\vec{x} = \dfrac{\vec{u}+\vec{v}}{|\vec{u}+\vec{v}|}$

But $|\vec{u}+\vec{v}|^2 = (\vec{u}+\vec{v}).(\vec{u}+\vec{v}) = 1+1+2\vec{u}.\vec{v} \quad [\because |\vec{u}| = |\vec{v}| = 1]$

$= 2+2\cos\alpha = 4\cos^2\alpha/2$

$\therefore \quad |\vec{u}+\vec{v}| = 2\cos\alpha/2 \Rightarrow \vec{x} = \frac{1}{2}(\sec\alpha/2)(\vec{u}+\vec{v})$

Similarly, $\vec{y} = \frac{1}{2}\sec\frac{\beta}{2}(\vec{v}+\vec{\omega})$ and $\vec{z} = \frac{1}{2}\sec\frac{\gamma}{2}(\vec{\omega}+\vec{u})$

Now consider $[\vec{x}\times\vec{y}\ \vec{y}\times\vec{z}\ \vec{z}\times\vec{x}]$

$= (\vec{x}\times\vec{y}).[(\vec{y}\times\vec{z})\times(\vec{z}\times\vec{x})]$

$= (\vec{x}\times\vec{y}).[\{(\vec{y}\times\vec{z}).\vec{x}\}\vec{z} - \{(\vec{y}\times\vec{z}).\vec{z}\}\vec{x}]$

[Using def^n of vector triple product.]

$= (\vec{x}\times\vec{y}).[[\vec{x}\,\vec{y}\,\vec{z}]\vec{z} - 0] \qquad [\because [\vec{y}\,\vec{z}\,\vec{z}] = 0]$

$= [\vec{x}\,\vec{y}\,\vec{z}][\vec{x}\,\vec{y}\,\vec{z}] = [\vec{x}\,\vec{y}\,\vec{z}]^2$ ...(1)

Also $[\vec{x}\,\vec{y}\,\vec{z}] = \left[\frac{1}{2}\left(\sec\frac{\alpha}{2}\right)(\vec{u}+\vec{v})\frac{1}{2}\sec\beta/2\right.$

$\left.(\vec{v}+\vec{\omega})\frac{1}{2}\sec\gamma/2(\vec{\omega}+\vec{u})\right]$

$$=\frac{1}{8}\sec\alpha/2\sec\beta/2\sec\gamma/2[\vec{u}+\vec{v}\ \vec{v}+\vec{\omega}\ \vec{\omega}+\vec{u}]$$

$$=\frac{1}{8}\sec\alpha/2\sec\beta/2\sec\gamma/2[(\vec{u}+\vec{v}).\{(\vec{v}+\vec{\omega})\times(\vec{\omega}+\vec{u})\}]$$

$$=\frac{1}{8}\sec\alpha/2\sec\beta/2\sec\gamma/2[(\vec{u}+\vec{v}).(\vec{v}\times\vec{\omega}+\vec{v}\times\overrightarrow{u+\omega}\times\vec{u})]$$

$$=\frac{1}{8}\sec\alpha/2\sec\beta/2\sec\gamma/2[\vec{u}.(\vec{v}\times\vec{\omega})+\vec{v}.(\vec{\omega}\times\vec{u})]$$

$(\because [\vec{a}\vec{b}\vec{c}]=0$ whenever any two vectors are same)

$$=\frac{1}{8}(\sec\alpha/2\sec\beta/2\sec\gamma/2)2[\vec{u}\vec{v}\vec{\omega}]$$

$$=\frac{1}{4}(\sec\alpha/2\sec\beta/2\sec\gamma/2)[\vec{u}\vec{v}\vec{\omega}]$$

$$\therefore [\vec{x}\vec{y}\vec{z}]^2=\frac{1}{16}[\vec{u}\vec{v}\vec{\omega}]^2\sec^2\alpha/2\ \sec^2\beta/2\ \sec^2\gamma/2]\ ...(2)$$

From (1) and (2),

$$[\vec{x}\times\vec{y}\ \vec{y}\times\vec{z}\ \vec{z}\times\vec{x}]=\frac{1}{16}[\vec{u}\vec{v}\vec{\omega}]^2\ \sec^2\frac{\alpha}{2}.\sec^2\frac{\beta}{2}\sec^2\frac{\gamma}{2}$$

Q. 25. If $\vec{a},\vec{b},\vec{c}$ and $\vec{d}$ **are distinct vectors such that** $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$ and $\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$.

Prove that $(\vec{a}-\vec{d}).(\vec{b}-\vec{c})\neq 0$ i.e. $\vec{a}.\vec{b}+\vec{d}.\vec{c}\neq\vec{d}.\vec{b}+\vec{a}.\vec{c}$ **(2004 - 2 Marks)**

Solution. Given that $\vec{a}\neq\vec{b}\neq\vec{c}\neq\vec{d}$

Such that $\vec{a}\times\vec{c}=\vec{b}\times\vec{d}$ ...(1)

$\vec{a}\times\vec{b}=\vec{c}\times\vec{d}$ ...(2)

To prove $(\vec{a}-\vec{d}).(\vec{b}-\vec{c})\neq 0$

Subtracting eq^n (2) from (1) we get

$$\vec{a}\times(\vec{c}-\vec{b})=(\vec{b}-\vec{c})\times\vec{d} \Rightarrow \vec{a}\times(\vec{c}-\vec{b})=\vec{d}\times(\vec{c}-\vec{b})$$

$$\Rightarrow \vec{a}\times(\vec{c}-\vec{b})-\vec{d}\times(\vec{c}-\vec{b})=0$$

$$\Rightarrow (\vec{a}-\vec{d})\times(\vec{c}-\vec{b})=0 \Rightarrow (\vec{a}-\vec{d})\,||\,(\vec{c}-\vec{b})$$

$[\because \vec{a}-\vec{d}\neq 0, \vec{c}-\vec{b}\neq 0$ as all distinct $]$

$\Rightarrow$ Angle between $\vec{a}-\vec{d}$ and $\vec{c}-\vec{b}$ is either 0 or 180°.

$$\Rightarrow (\vec{a}-\vec{d}).(\vec{c}-\vec{b})=|\vec{a}-\vec{d}||\vec{c}-\vec{b}|\cos 0\,[\text{or } \cos 180^\circ]\neq 0$$

as a, b, c, d all are different. Hence Proved.

Q. 26. Find the equation of plane passing through (1, 1, 1) & parallel to the lines L1, L2 having direction ratios (1,0, –1), (1, –1,0). Find the volume of tetrahedron formed by origin and the points where these planes intersect the coordinate axes. (2004 - 2 Marks)

Ans. $x+y+z=3\,;\,\frac{9}{2}$ cubic units

Solution. $\therefore$ The plane is parallel to the lines L_1 and L_2 with direction ratios as (1, 0, –1) and (1, –1, 0)

$\therefore$ A vector perpendicular to L_1 and L_2 will be parallel to the normal $(\vec{n})$ to the plane.

$$\therefore \vec{n}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 0 & -1\\ 1 & -1 & 0\end{vmatrix}=-\hat{i}-\hat{j}-\hat{k}$$

$\therefore$ Eqn. of plane through (1, 1, 1) and having normal vector $\vec{n}=-\hat{i}-\hat{j}-\hat{k}$ is given by $(\vec{r}-\vec{a}).\vec{n}=0$

$$\Rightarrow -1(x-1)-1(y-1)-1(z-1)=0 \Rightarrow x+y+z=3$$

$$\Rightarrow \frac{x}{3}+\frac{y}{3}+\frac{z}{3}=1 \qquad ...(1)$$

Now the pts where this plane meets the axes are A(3, 0, 0), B(0, 3, 0),C (0, 0, 3)

$\therefore$ Vol.of tetrahedron OABC

$$=\frac{1}{6}\times \text{Area of base}\times \text{altitude}$$

$$=\frac{1}{6}\times \text{Ar}(\Delta ABC)\times \text{length of } \perp^{\text{lar}}\ (0,0,0) \text{ to plane (1)}$$

$$=\frac{1}{6}\times\frac{1}{2}\left[\frac{\sqrt{3}}{4}\times|\overrightarrow{AB}|^2\right]\times\left[\left|\frac{-3}{\sqrt{1+1+1}}\right|\right]$$

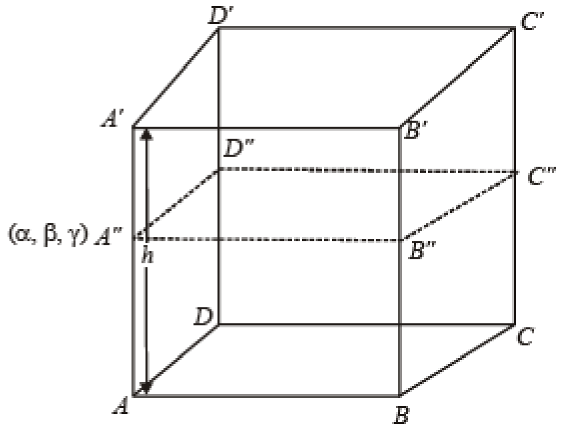
(Note that ΔABC is an equilateral Δ here.)

$$=\frac{1}{12}\times\frac{\sqrt{3}}{4}\times(3\sqrt{2})^2\times\sqrt{3}\ =\frac{3\times 18}{48}=\frac{9}{2}\ \text{cubic units.}$$

Q. 27. A parallelepiped 'S' h as base points A, B, C an d D and upper face points A', B', C' and D'. This parallelepiped is compressed by upper face A'B'C'D' to form a new parallelepiped 'T' having upper face points A'', B'', C'' and D''. Volume of parallelepiped T is 90 percent of the volume of parallelepiped S. Prove that the locus of 'A''', is a plane. (2004 - 2 Marks)

Solution. ATQ 'S' is the parallelepiped with base points A, B, C and D and upper face points A', B',C' and D ' . Let its vol. be V_s .

By compressing it by upper face A', B', C ',D', a new parallelepiped 'T' is formed whose upper face pts are now A",B",C" and D ". Let its vol. be V_T .



Let h be the height of original parallelepiped S.

Then V_S = (ar ABCD)x h ...(1)

Let equation of plane ABCD be ax + by +cz +d= and A"(α,β,γ)

Then height of new parallelepiped T is the length of perpendicular from A" to ABCD

i.e. $\dfrac{a\alpha+b\beta+c\gamma+d}{\sqrt{a^2+b^2+c^2}}$

$$\therefore \quad V_T=(ar\,ABCD)\times\frac{a\alpha+b\beta+c\gamma+d}{\sqrt{a^2+b^2+c^2}} \qquad ...(2)$$

But given that,

$$V_T=\frac{90}{100}V_s \qquad ...(3)$$

From (1), (2) and (3) we get,

$$\frac{a\alpha+b\beta+c\gamma+d}{\sqrt{a^2+b^2+c^2}}=0.9h$$

$$\Rightarrow \quad a\alpha+b\beta+c\gamma+(d-0.9h\sqrt{a^2+b^2+c^2})=0$$

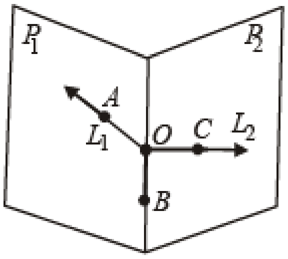
$\therefore$ Locus of $A''(\alpha,\beta,\gamma)$ is

$$ax+by+cz+(d-0.9h\sqrt{a^2+b^2+c^2})=0$$

Which is a plane parallel to ABCD . Hence proved.

Q. 28. P_1 and P_2 are planes passing through origin. L_1 and L_2 are two line on P_1 and P_2respectively such that their intersection is origin. Show that there exists points A, B, C, whose permutation A', B', C' can be chosen such that (i) A is on L_1, B on P_1 but not on $L_1$and C not on P_1 (ii) A' is on L_2, B' on P_2 but not on L_2 and C' not on P_2 (2004 - 4 Marks)

Solution. Following fig. shows the possible situation for planes P_1 and P_2 and the lines L_1 and L_2



Now if we choose pts A, B, C as follows.

A on L_1 , B on the line of intersection of P_1 and P_2 but other than origin and C on L_2 again other than origin then we can consider

A corresponds to one of A',B',C' and

B corresponds to one of the remaining of A', B',C' and

C corresponds to third of A',B',C' e.g.

$A' \equiv C; B' \equiv B; C' \equiv A$

Hence one permutation of [ABC] is [CBA]. Hence Proved.

Q. 29. Find the equation of the plane containing the line 2x – y + z – 3 = 0, 3x + y + z = 5 and at a distance of $\frac{1}{\sqrt{6}}$ from the point (2, 1, –1) (2005 - 2 Marks)

Ans. 62 x + 29 y + 19z - 105 = 0

Solution. The given line is 2x - y + z - 3 = 0 = 3x + y + z - 5

Which is intersection line of two planes

2 x - y + z - 3 = 0 ...(i)

and 3x + y + z - 5 = 0 ...(ii)

Any plane containing this line will be the plane passing through the intersection of two planes (i) and (ii).

Thus the plane containing given line can be written as

$$(2x - y + z - 3) + \lambda(3x + y + z - 5) = 0$$

$$\Rightarrow \quad (3\lambda + 2)x + (\lambda - 1)y + (\lambda + 1)z + (-5\lambda - 3) = 0$$

As its distance from the pt. $(2, 1, -1)$ is $\frac{1}{\sqrt{6}}$

$$\therefore \quad \left|\frac{(3\lambda+2)2+(\lambda-1)1+(\lambda+1)(-1)+(-5\lambda-3)}{\sqrt{(3\lambda+2)^2+(\lambda-1)^2+(\lambda+1)^2}}\right| = \frac{1}{\sqrt{6}}$$

$$\Rightarrow \quad \left|\frac{\lambda-1}{\sqrt{11\lambda^2+12\lambda+6}}\right| = \frac{1}{\sqrt{6}}$$

Squaring both sides, we get

$$\frac{(\lambda-1)^2}{11\lambda^2+12\lambda+6} = \frac{1}{6}$$

$$\Rightarrow \quad 6\lambda^2-12\lambda+6-11\lambda^2-12\lambda-6=0$$

$$\Rightarrow \quad 5\lambda^2+24\lambda=0 \Rightarrow \lambda(5\lambda+24)=0$$

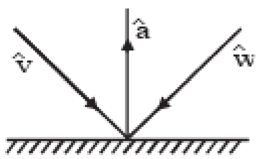
$$\Rightarrow \quad \lambda = 0 \text{ or } -24/5$$

∴ The required equations of planes are 2x- y+z-3 = 0

$$\text{and}\left[3\left(\frac{-24}{5}\right)+2\right]x+\left[-\frac{24}{5}-1\right]y \quad +\left[-\frac{24}{5}+1\right]z-5\left(\frac{-24}{5}\right)-3=0$$

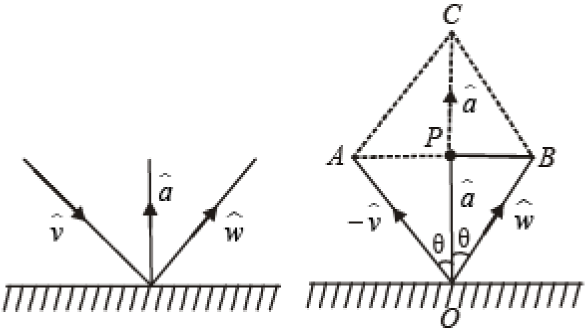
or 62x + 29 y + 19z - 105 = 0

Q. 30. If the incident ray on a surface is along the unit vector $\hat{v}$, the reflected ray is along the unit vector $\hat{w}$ and the normal is along unit vector $\hat{a}$ outwards. Express $\hat{w}$ in in terms of $\hat{a}$ and $\hat{v}$. (2005 - 4 Marks)



Ans. $\hat{\omega} = \hat{v} - 2(\hat{a}\cdot\hat{v})\hat{a}$

Solution. Given that incident ray is along $\hat{v}$, reflected ray is along $\hat{w}$ and normal is along $\hat{a}$, outwards. The given figure can be redrawn as shown.



We know that incident ray, reflected ray and normal lie in a plane, and angle of incidence = angle of reflection.

Therefore $\hat{a}$, will be along the angle bisector of $\hat{w}$ and $-\hat{v}$,

i.e., $\hat{a} = \dfrac{\hat{w} + (-\hat{v})}{|\hat{w} - \hat{v}|}$ ...(1)

[∵ Angle bisector will along a vector dividing in same ratio as the ratio of the sides forming that angle.]

But $\hat{a}$ is a unit vector.

Where $|\hat{w} - \hat{v}| = OC = 2OP = 2|\hat{w}|\cos\theta = 2\cos\theta$

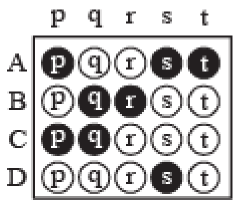
Substituting this value in equation (1) we get

$$\hat{a} = \frac{\hat{w} - \hat{v}}{2\cos\theta}$$

$\therefore \quad \hat{w} = \hat{v} + (2\cos\theta)\hat{a} = \hat{v} - 2(\hat{a}.\hat{v})\hat{a} \quad [\because \hat{a}.\hat{v} = -\cos\theta]$

Match the Following of Vector Algebra & 3D (Part - 1) Geometry

DIRECTIONS (Q. 1-6) : Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example :
If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



Q. 1. Match the following :

(A) Two rays x + y = |a| and ax – y = 1 intersects each other in the **(p)**
2
first quadrant in the interval a ∈ (a_0, ∞), the value of a_0 is

(B) Point (α , β , γ) lies on the plane x + y + z = 2. **(q)**
4/3

Let $\vec{a} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$, $\hat{k} \times (\hat{k} \times \vec{a}) = 0$, then $\gamma =$

(C) $\left|\int_0^1 (1-y^2)\,dy\right| + \left|\int_1^0 (y^2-1)\,dy\right|$ **(r)**

$\left|\int_0^1 \sqrt{1-x}\,dx\right| + \left|\int_{-1}^0 \sqrt{1+x}\,dx\right|$

(D) If sinA sinB sinC + cosA cosB = 1, then the value of sinC = **(s)**
1

Ans. (A) → s; (B) → p; (C) → r, q; (D) → s

Solution. (A) On solving the given equations x + y = |a| and ax – y = 1, we get

$x = \frac{1+|a|}{a+1}$ and $y = \frac{a|a|-1}{a+1}$

∴ Rays intersect each other in I quad.

$\therefore x, y > 0 \Rightarrow a+1 > 0$ and $a|a| - 1 > 0 \Rightarrow a > 1$
$\therefore a_0 = 1 (A) \rightarrow (s)$

(B) (α, β, γ) lies on the plane x + y + z = 2

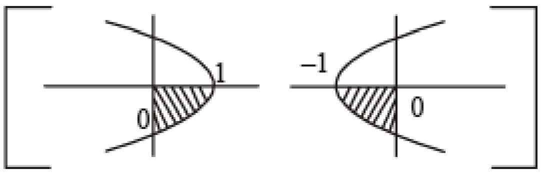
⇒ α + β + γ = 2

Also $\hat{k} \times (\hat{k} \times \vec{a}) = (\hat{k}.\hat{a})\hat{k} - (\hat{k}.\hat{k})\vec{a}$

$\Rightarrow \gamma\hat{k} - \alpha\hat{i} - \beta\hat{j} - \gamma\hat{k} = 0 \Rightarrow \alpha\hat{i} + \beta\hat{j} = 0$
$\Rightarrow \alpha = 0 = \beta \Rightarrow \gamma = 2 \qquad (\because \alpha + \beta + \gamma = 2)$
(B) → (p)

(C) $\left|\int_0^1 (1-y^2)dy\right| + \left|\int_0^1 (y^2-1)dy\right| = 2\int_0^1 (1-y^2)dy = \frac{4}{3}$

Also, $\left|\int_0^1 \sqrt{1-x}\,dx\right| + \int_{-1}^0 \sqrt{1+x}\,dx = 2\int_0^1 \sqrt{1-x}\,dx$



$[\because y = \sqrt{1-x}$, i.e., $y^2 = -(x-1)$ and $y = \sqrt{1+x}$

i.e., $y^2 = (x + 1)$ represent same area under the given limits]

$= 2\int_0^1 \sqrt{x}\,dx \left[\text{Using} \int_0^a f(x)dx = \int_0^a f(a-x)dx\right]$

$= \left[2.\frac{2}{3}x^{3/2}\right]_0^1 = \frac{4}{3}$, (C) → (r) and (q)

(D) Given : sin A sin B sin C + cos A cos B = 1

But sin A sin B sin C + cos A cos B ≤ sin A sin B + cos

A cos B = cos (A – B)

$\Rightarrow \cos(A - B) \geq 1 \;\; \Rightarrow \cos(A - B) = 1$

$\Rightarrow A - B = 0 \Rightarrow A = B$

$\therefore$ Given relation becomes $\sin^2 A \sin C + \cos^2 A = 1$

$\Rightarrow \sin C = 1,$

(D) → (s)

Q. 2. Consider the following linear equations ax + by + cz = 0; bx + cy + az = 0; cx + ay + bz = 0

Match the conditions/expressions in Column I with statements in Column II and indicate your answer by darkening the appropriate bubbles in the 4 × 4 matrix given in the ORS.

Column I	Column II
(A) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	**(p) the equations represent planes meeting only at asingle point**
(B) $a + b + c = 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	**(q) the equations represent the line x = y = z.**
(C) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 \neq ab + bc + ca$	**(r) the equations represent identical planes.**
(D) $a + b + c = 0$ and $a^2 + b^2 + c^2 = ab + bc + ca$	**(s) the equations represent the whole of the three dimensional space.**

Ans. (A) → r; (B) → q; (C) → p; (D) → s

Solution. Here we have, the determinant of the coefficient matrix of given equation, as

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= -\frac{1}{2}(a+b+c)[(a-b)^2+(b-c)^2+(c-a)^2]$$

(A) $a+b+c \neq 0$ and $a^2+b^2+c^2-ab-bc-ca=0$

$\Rightarrow \;\; (a-b)^2+(b-c)^2+(c-a)^2=0$

$\Rightarrow \;\; a=b=c \quad (\text{but} \neq 0 \text{ as } a+b+c \neq 0)$

This equation represent identical planes.

(B) a +b +c = 0 and $a^2 + b^2 + c^2 - ab - bc - ca \neq 0$

$\Rightarrow \Delta = 0$ and a, b, c are not all equal.

∴ All equations are not identical but have infinite many solutions.

∴ ax + by = (a+ b)z (using a+b+c = 0) and bx + cy = (b+ c)z

$\Rightarrow (b^2 - ac)\, y = (b^2 - ac)z \Rightarrow y = z$

$\Rightarrow ax + by + cy = 0 \Rightarrow ax = ay \Rightarrow x = y$

$\Rightarrow x = y = z$

∴ The equations represent the line x = y = z

(C) $a + b + c \neq 0$ and $a^2 + b^2 + c^2 - ab - bc - ca \neq 0$

$\Rightarrow \Delta \neq 0 \Rightarrow$ Equations have only trivial solution i.e., x = y = z = 0

∴ the equations represents the three planes meeting at a single point namely origin.

(D) a +b +c= 0 and $a^2 + b^2 + c^2 - ab - bc - ca = 0$

$\Rightarrow a = b = c$ and $\Delta = 0 \Rightarrow a = b = c = 0$

⇒ All equations are satisfied by all x, y, and z.

⇒ The equations represent the whole of the three dimensional space (all points in 3–D)

Q. 3. Match the statements / expressions given in Column-I with the values given in Column-II.

Column-I	**Column-II**
(A) Root(s) of the equation $2\sin^2\theta + \sin^2 2\theta = 2$	**(p) π/6**
(B) Points of discontinuity of the unction $f(x) = \left[\frac{6x}{\pi}\right]\cos\left[\frac{3x}{\pi}\right]$, **f where [y] denotes the largest integer less than or equal to y**	**(q) π/4**
(C) Volume of the parallelopiped with its edges represented by the vectors $\hat{i}+\hat{j}, \hat{i}+2\hat{j}$ and $\hat{i}+\hat{j}+\pi\hat{k}$	**(r) π/3**

(D) Angle between vector $\vec{a}$ and $\vec{b}$ where $\vec{a}$, $\vec{b}$ and $\vec{c}$ are unit vectors **(s) π/2**

satisfying $\vec{a}+\vec{b}+\sqrt{3}\,\vec{c}=\vec{0}$

(t) π

Ans. A → q,s; B → p,r,s, t; C → t; D→ r

Solution.

(A) The given equation is

$$2\sin^2\theta+\sin^2 2\theta=2$$

$$\Rightarrow\ 2\sin^2\theta+4\sin^2\theta\cos^2\theta-2=0$$

$$\Rightarrow\ \sin^2\theta+2\sin^2\theta(1-\sin^2\theta)-1=0$$

$$\Rightarrow\ 2\sin^4\theta-3\sin^2\theta+1=0$$

$$\Rightarrow\ 2\sin^4\theta-2\sin^2\theta-\sin^2\theta+1=0$$

$$\Rightarrow\ 2\sin^2\theta(\sin^2\theta-1)-1(\sin^2\theta-1)=0$$

$$\Rightarrow\ (\sin^2\theta-1)(2\sin^2\theta-1)=0$$

$$\Rightarrow\ \sin^2\theta=1 \text{ or } \sin^2\theta=\frac{1}{2}$$

$$\Rightarrow\ \sin^2\theta=\sin^2\frac{\pi}{2} \text{ or } \sin^2\theta=\sin^2\frac{\pi}{4}$$

$$\Rightarrow\ \theta=n\pi\pm\frac{\pi}{2} \text{ or } n\pi\pm\frac{\pi}{4}\ \Rightarrow\ \theta=\frac{\pi}{2} \text{ or } \frac{\pi}{4}$$

(B) We know that [x] is discontinuous at all integral

values, therefore $\left[\frac{6x}{\pi}\right]$ is discontinuous at $x=\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}$ and π. Also cos $\left[\frac{3x}{\pi}\right]\neq 0$ for any of

these values of x.

$\therefore$ $\left[\frac{6x}{\pi}\right]\cos\left[\frac{3x}{\pi}\right]$ is discontinuous at $x=\frac{\pi}{6},\frac{\pi}{3},\frac{\pi}{2}$ and π.

(C) We know that th e volume of a parallelopipe with coterminous edges as $\vec{a},\vec{b}$ and $\vec{c}$ is

given by $[\vec{a}\ \vec{b}\ \vec{c}]$

∴ The required volume is $=\vec{a}.\vec{b}\times\vec{c}$

$$=\begin{vmatrix}1 & 1 & 0\\1 & 2 & 0\\1 & 1 & \pi\end{vmatrix}=\pi$$

(D) We have $\vec{a}+\vec{b}=-\sqrt{3}\,\vec{c} \Rightarrow |\vec{a}+\vec{b}|^2=3|\vec{c}|^2$

$\Rightarrow (\vec{a}+\vec{b}).(\vec{a}+\vec{b})=3\vec{c}.\vec{c}$

$\Rightarrow \vec{a}.\vec{a}+\vec{b}.\vec{b}+2\vec{a}.\vec{b}=3\vec{c}.\vec{c} \Rightarrow 1+1+2\cos\theta=3$

(Where θ is the angle between $\vec{a}$ and $\vec{b}$)

$\Rightarrow \cos\theta=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$

Q. 4. Match the statements/expressions given in Column-I with the values given in Column-II.

Column-I	Column-II
(A) The number of solutions of the equation $x\,e^{\sin x}-\cos x=0$ in the interval $\left(0,\frac{\pi}{2}\right)$	**(p) 1**
(B) Value(s) of k for which the planes k x + 4y + z = 0, 4x + k y + 2z = 0 and 2x + 2y + z = 0 intersect in a straight line	**(q) 2**
(C) Value(s) of k for which | x – 1 | + | x – 2 | + | x + 1 | | + | x + 2 | = 4k has integer solution(s)	**(r) 3**
(D) If y' = y + 1 and y (0) = 1, then value(s) of y (1n 2)	**(s) 4**
	(t) 5

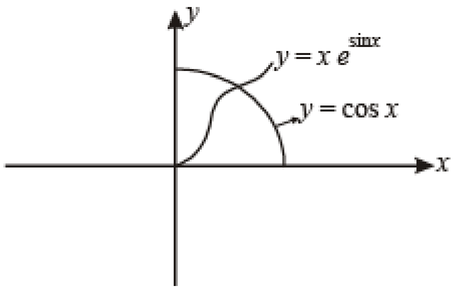
Ans. A → p;B → q, s;C → q, r, s, t;D→ r

Solution. (A) For the solution of $x\,e^{\sin x}-\cos x=0$ in $\left(0,\frac{\pi}{2}\right)$

Let us consider two functions

$y=x\,e^{\sin x}$ and $y=\cos x$

The range of $y=xe^{\sin x}$ is $\left(0,\frac{\pi e}{2}\right)$, also it is an increasing function on $\left(0,\frac{\pi}{2}\right)$. Their graph are as shown in the figure below :



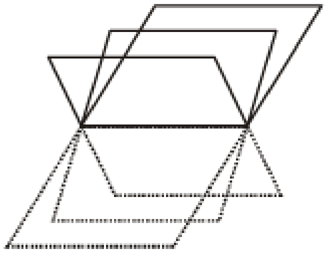
Clearly the two curves meet only at one point, therefore the given equation has only one solution in $\left(0, \frac{\pi}{2}\right)$.

(B) Three given planes are

kx + 4 y +z = 0

4x + ky + 2z = 0

2x + 2 y +z = 0



Clearly all the planes pass through (0,0,0).

∴ Their line of intersection also pass through (0, 0, 0) Let a, b, c, be the direction ratios of required line, then we should have

ka + 4b +c = 0

4a + kb + 2c = 0

2a + 2b +c = 0
For the required line to exist the above system of equations in a, b, c, should have non trivial solution i.e.

$$\begin{vmatrix} k & 4 & 1 \\ 4 & k & 2 \\ 2 & 2 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad k(k-4)-4(4-4)+1(8-2k)=0$

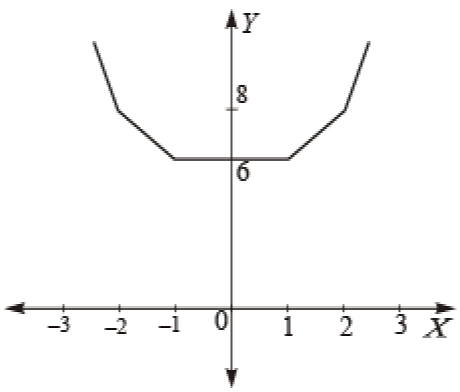
$\Rightarrow \quad k^2-6k+8=0 \Rightarrow (k-2)(k-4)=0$

$\Rightarrow \quad k=2 \text{ or } 4$

(C) We have $f(x)=|x-1|+|x-2|+|x+1|+|x+2|$

$$=\begin{cases} -4x \quad , & x \le -2 \\ -2x+4 , & -2 < x \le -1 \\ 6 , & -1 < x \le 1 \\ 2x+4 , & 1 < x \le 2 \\ 4x \quad , & x \ge 2 \end{cases}$$

The graph of the above function is as given below



Clearly, from graph, $f(x) \ge 6$

$\Rightarrow \quad 4k \ge 6 \quad \Rightarrow \quad k \ge \frac{3}{2}$

$\therefore \quad k=2, 3, 4, 5, 6, \ldots\ldots$

(D) Given that

$\frac{dy}{dx} = y+1$ and $y(0)=1$

$\Rightarrow \int \frac{dy}{y+1} = \int dx \Rightarrow \ln|y+1| = x+c$

At $x=0, y=1 \Rightarrow c = ln\,2$

$\therefore \ln|y+1| = x + ln\,2 \Rightarrow y+1 = 2e^x \Rightarrow y = 2e^x - 1$

$\therefore \quad y(\ln 2) = 2e^{\ln 2} - 1 = 2\times 2 - 1 = 3$

Q. 5. Match the statement in Column-1 with the values in Column –II

Column – I **Column – II**

(A) A line from the origin meets the lines $\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+1}{1}$ **(p) – 4**

$\frac{x-\frac{8}{3}}{2} = \frac{y+3}{-1} = \frac{z-1}{1}$ **and at P and Q respectively..**

If length PQ = d, then d^2 is

(B) The values of x satisfying **(q) 0**

$\tan^{-1}(x+3) - \tan^{-1}(x-3) = \sin^{-1}\left(\frac{3}{5}\right)$ are

(C) Non-zero vectors $\vec{a}, \vec{b}$ and $\vec{c}$ satisfy $\vec{a}.\vec{b} = 0$. **(r) 4**

$(\vec{b}-\vec{a}).(\vec{b}+\vec{c}) = 0$ and $2|\vec{b}+\vec{c}| = |\vec{b}-\vec{a}|$.

If $\vec{a} = \mu\vec{b} + 4\vec{c}$, **, then the possible values of μ are**

(D) Let f be the function on [–π, π] given by f (0) = 9 **(s) 5**

and $f(x) = \sin\left(\frac{9x}{2}\right) \Big/ \sin\left(\frac{x}{2}\right) for\ x \neq 0$

The value of $\frac{2}{\pi}\int_{-\pi}^{\pi} f(x)dx$ is

Ans. (A) → t; (B) → p, r; (C) → q, s; (D) → r

Solution. Let the line through origin be $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$ — (1)
then as it intersects

$\frac{x-2}{1} = \frac{y-1}{-2} = \frac{z+1}{1}$ —(2)

and $\frac{x-8/3}{2} = \frac{y+3}{-1} = \frac{z-1}{1}$ —(3)

at P and Q, shortest distance of (1) with (2) and (3) should be zero.

$$\therefore \text{Using} \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

$$\text{we get} \begin{vmatrix} 2 & 1 & -1 \\ a & b & c \\ 1 & -2 & 1 \end{vmatrix} = 0 \Rightarrow a+3b+5c=0 \quad \text{—(4)}$$

$$\text{and} \begin{vmatrix} 8/3 & -3 & 1 \\ a & b & c \\ 2 & -1 & 1 \end{vmatrix} = 0 \Rightarrow 3a+b-5c=0 \quad \text{—(5)}$$

Solving (4) and (5), we get

$$\frac{a}{-15-5} = \frac{b}{15+5} = \frac{c}{1-9} \text{ or } \frac{a}{5} = \frac{b}{-5} = \frac{c}{2}$$

Hence equation (1) becomes $\frac{x}{5} = \frac{y}{-5} = \frac{z}{2} = \lambda$

For some value of λ, $P(5\lambda, -5\lambda, 2\lambda)$

Which lies on (2) also

$$\therefore \frac{5\lambda-2}{1} = \frac{-5\lambda-1}{-2} = \frac{2\lambda+1}{1} \Rightarrow \lambda = 1$$

$$\therefore P(5, -5, 2)$$

Also for some value of λ, $Q(5\lambda, -5\lambda, 2\lambda)$

Which lies on (3) also

$$\therefore \frac{5\lambda-8/3}{3} = \frac{-5\lambda+3}{-1} = \frac{2\lambda-1}{1} \Rightarrow \lambda = 2/3$$

$$\therefore Q\left(\frac{10}{3}, \frac{-10}{3}, \frac{4}{3}\right)$$

Hence $d^2 = PQ^2 = \left(\frac{25}{9}+\frac{25}{9}+\frac{4}{9}\right) = 6$

(B) $\tan^{-1}(x+3) - \tan^{-1}(x-3) = \tan^{-1}\frac{3}{4}$

$\Rightarrow \tan^{-1}\left(\frac{x+3-x+3}{1+x^2-9}\right) = \tan^{-1}\left(\frac{3}{4}\right),\ x^2-9 \geq -1$

$\Rightarrow \frac{6}{x^2-8} = \frac{3}{4} \Rightarrow x^2 = 16$ or $x = 4, -4$

(C) We have $\vec{c} = \frac{\vec{a}-\mu\vec{b}}{4}$

Then $(\vec{b}-\vec{a}).(\vec{b}+\vec{c}) = 0$

$\Rightarrow (\vec{b}-\vec{a}).\left(\vec{b}+\frac{\vec{a}-\mu\vec{b}}{4}\right) = 0$

$\Rightarrow (\vec{b}-\vec{a}).\left(\frac{4-\mu}{4}\vec{b}+\frac{\vec{a}}{4}\right) = 0 \Rightarrow \frac{4-\mu}{4}b^2 - \frac{a^2}{4} = 0$

From (1) and (2), we get

$\frac{(4-\mu)^2-4}{4-\mu} = \frac{3}{1}$

$\Rightarrow \mu^2 - 8\mu + 12 = 12 - 3\mu \Rightarrow \mu^2 - 5\mu = 0$

$\Rightarrow \mu = 0$ or 5

(D) $I = \frac{2}{\pi}\int_{-\pi}^{\pi} \frac{\sin(9x/2)}{\sin(x/2)}dx = \frac{2}{\pi} \times 2\int_0^{\pi} \frac{\sin 9x/2}{\sin x/2}dx$

Let $\frac{x}{2} = 0 \Rightarrow dx = 2d\theta$

Also at $x = 0, \theta = 0$ and at $x = x, \theta = \pi/2$

$\therefore I = \frac{8}{\pi}\int_0^{\pi/2} \frac{\sin 9\theta}{\sin\theta}d\theta$

$I = \frac{8}{\pi}\int_0^{\pi/2}\left[\frac{\sin 9\theta - \sin 7\theta}{\sin\theta} + \frac{(\sin 7\theta - \sin 5\theta)}{\sin\theta} + \right.$

$\left.\frac{(\sin 5\theta - \sin 3\theta)}{\sin\theta} + \frac{(\sin 3\theta - \sin\theta)}{\sin\theta} + \frac{\sin\theta}{\sin\theta}\right]d\theta$

$$= \frac{16}{\pi}\int_0^{\pi/2}(\cos 8\theta + \cos 6\theta + \cos 4\theta + \cos 2\theta + 1)\,d\theta + \frac{8}{\pi}\int_0^{\pi/2} d\theta$$

$$= \frac{16}{\pi}\left[\frac{\sin 8\theta}{8} + \frac{\sin 6\theta}{6} + \frac{\sin 4\theta}{4} + \frac{\sin 2\theta}{2} + \theta\right]_0^{\pi/2} + \frac{8}{\pi}(\theta)_0^{\pi/2}$$

$$= 0 + \frac{8}{\pi}\left(\frac{\pi}{2} - 0\right) = 4$$

Q. 6. Match the statements given in Column-I with the values given in Column-II.

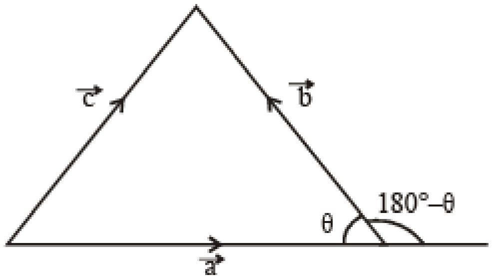
Column-I	**Column-II**
(A) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}$, $\vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ **form a triangle, then the internal angle of the triangle between** $\vec{a}$ and $\vec{b}$	**(p) π/6**
(B) If $\int_a^b (f(x) - 3x)\,dx = a^2 - b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	**(q) 2π/3**
(C) The value of $\frac{\pi^2}{\ell n 3}\int_{7/6}^{5/6} \sec(\pi x)\,dx$ is	**(r) π/3**
(D) The maximum value of $\left\|Arg\left(\frac{1}{1-z}\right)\right\|$ for $\|z\| = 1, z \neq 1$ **is given by**	**(s) π**
	(t) π/2

Ans. A→q, B→p, C→s, D→t

Solution. As $\vec{a} + \vec{b} = \vec{c}$

∴ The figure is as shown.

Clearly $\cos(180 - \theta) = \frac{\vec{a}\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{2}$



$\Rightarrow \cos\theta = -\frac{1}{2} \Rightarrow \theta = \frac{2\pi}{3}$

$\Rightarrow$ **A $\rightarrow$ q**

$\int_a^b (f(x) - 3x)\,dx = a^2 - b^2$

$\Rightarrow \int_a^b f(x)\,dx + \frac{3}{2}[-b^2 + a^2] = a^2 - b^2$

$\Rightarrow \int_a^b f(x)\,dx = -\frac{1}{2}(a^2 - b^2)$

$\Rightarrow \frac{d}{db}\left[\int_a^b f(x)\,dx\right] = b \Rightarrow f(b) = b \Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$

$\Rightarrow$ **B $\rightarrow$ p**

$\frac{\pi^2}{\ell n3}\int_{\frac{7}{6}}^{\frac{5}{6}} \sec(\pi x)dx = \frac{\pi^2}{\pi \ell n3}[\ell n|\sec \pi x + \tan \pi x|]_{\frac{7}{6}}^{\frac{5}{6}}$

$= \frac{\pi}{\ell n3}\left[\ell n|\sec\frac{5\pi}{6} + \tan\frac{5\pi}{6} - \ell n|\sec\frac{7\pi}{6} + \tan\frac{7\pi}{6}|\right]$

$= \frac{\pi}{\ell n3}\left[\ell n\left|-\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right| - \ell n\left|-\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right|\right] = \frac{\pi}{\ell n3}\ell n3 = \pi$

$\therefore$ **C $\rightarrow$ s**

For $|z| = 1$ and $z \neq 1$. Let $z = e^{i\theta}$

Then $1 - z = 1 - \cos\theta - i\sin\theta = 2\sin^2\frac{\theta}{2} - 2i\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

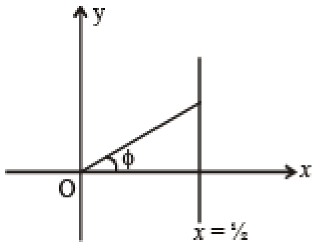
or $1 - z = 2\sin\frac{\theta}{2}\left[\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right]$

$\therefore \frac{1}{1-z} = \frac{1}{2}\left[1 + i\cot\frac{\theta}{2}\right]$

Here real part of $\frac{1}{1-z}$ is always $\frac{1}{2}$

$\therefore$ Locus of $\frac{1}{1-z}$ is $x = \frac{1}{2}$

For which max $\left|Arg\left(\frac{1}{1-z}\right)\right|$ is max. value of ϕ i.e. $\frac{\pi}{2}$.



Clearly max. $\left|Arg\left(\frac{1}{1-z}\right)\right|$ approaches to $\frac{\pi}{2}$ but will not be attained.

$\therefore$ D $\rightarrow$ t.

Match the Following of Vector Algebra & 3D (Part - 2) Geometry

DIRECTIONS (Q. 7-9) : Each question has matching lists. The codes for the lists have choices (a), (b), (c) and (d) out of which ONLY ONE is correct.

Q. 7. Match List I with List II and select the correct answer using the code given below the lists :

List I | **List II**

P. Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 2. **Then the volume of the parallelepiped determined by vectors** $2(\vec{a}\times\vec{b}), 3(\vec{b}\times\vec{c})$ and $2(\vec{c}\times\vec{a})$ is | **1. 100**

Q. Volume of parallelepiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 5. **Then the volume of the parallelepiped determined by vectors** $3(\vec{a}+\vec{b}), 3(\vec{b}+\vec{c})$ and $2(\vec{c}+\vec{a})$ is | **2. 30**

R. Area of a triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. **Then the area of the triangle with adjacent sides determined by vectors** $(2\vec{a}+3\vec{b})$ and $(\vec{a}-\vec{b})$ is | **3. 24**

S. Area of a parallelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. **Then the area of the parallelogram with adjacent sides determined by vectors** $(\vec{a}+\vec{b})$ and $\vec{a}$ is | **4. 60**

Codes:

	P	Q	R	S
(a)	4	2	3	1
(b)	2	3	1	4
(c)	3	4	1	2
(d)	1	4	3	2

Ans. (c)

Solution. (P) $[\vec{a}\ \vec{b}\ \vec{c}] = 2$

$\therefore\ [2(\vec{a}\times\vec{b})\ \ 3(\vec{b}\times\vec{c})\ \ \vec{c}\times\vec{a}]$

$= 6[\vec{a}\times\vec{b}\ \ \vec{b}\times\vec{c}\ \ \vec{c}\times\vec{a}]$

$= 6[\vec{a}\ \vec{b}\ \vec{c}]^2 = 6\times 4 = 24$

$\therefore$ (P) $\to$ (3)

(Q) $[\vec{a}\ \vec{b}\ \vec{c}] = 5$

$\therefore\ [3(\vec{a}+\vec{b})\ \ \vec{b}+\vec{c}\ \ 2(\vec{c}+\vec{a})]$

$= 6[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]$

$= 6\times 2[\vec{a}\ \vec{b}\ \vec{c}] = 6\times 2\times 5 = 60$

$\therefore$ (Q) $\to$ (4)

(R) $\frac{1}{2}|\vec{a}\times\vec{b}| = 20 \Rightarrow |\vec{a}\times\vec{b}| = 40$

$\therefore\ \frac{1}{2}|(2\vec{a}+3\vec{b})\times(\vec{a}-\vec{b})| = \frac{1}{2}|-2\vec{a}\times\vec{b}+3\vec{b}\times\vec{a}|$

$= \frac{1}{2}\times 5|\vec{a}\times\vec{b}| = \frac{5}{2}\times 40 = 100$

$\therefore$ (R) $\to$ (1)

(S) $|\vec{a}\times\vec{b}| = 30$

$\therefore\ |(\vec{a}+\vec{b})\times\vec{a}| = |(\vec{b}\times\vec{a})| = 30$

$\therefore$ (s) $\to$ (2)

Q. 8. Consider the lines $L_1: \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}, L_2\ \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$ **and the planes P_1 : 7x + y + 2z = 3, P_2 = 3x + 5y – 6z = 4. Let ax + by + cz = d be the equation of the plane passing through the point of intersection of lines L_1 and L_2, and perpendicular to planes P_1 and P_2.**

Match List I wish List II and select the correct answer using the code given below the lists :

	List I	List II	
P.	$a =$	1.	13
Q.	$b =$	2.	-3
R.	$c =$	3.	1
S.	$d =$	4.	-2

Codes:

	P	Q	R	S
(a)	3	2	4	1
(b)	1	3	4	2
(c)	3	2	1	4
(d)	2	4	1	3

Ans. (a)

Solution. Any point on L_1 is $(2\lambda + 1, -\lambda, \lambda - 3)$ and that on L_2 is $(\mu + 4, \mu - 3, 2\mu - 3)$ For point of intersection of L_1 and L_2 $2\lambda + 1 = \mu + 4, -\lambda = \mu - 3, \lambda - 3 = 2\mu - 3 \Rightarrow \lambda = 2, \mu = 1$

$\therefore$ Intersection point of L_1 and L_2 is $(5, -2, -1)$

$\because$ $ax + by + cz = d$ is perpendicular to p_1 & p_2

$\therefore$ $7a + b + 2c = 0$ and $3a + 5b - 6c = 0$

$$\Rightarrow \frac{a}{-16} = \frac{b}{48} = \frac{c}{32} \Rightarrow \frac{a}{1} = \frac{b}{-3} = \frac{c}{-2}$$

$\therefore$ Equation of plane is $x - 3y - 2z = d$

As it passes through $(5, -2, -1)$

$\therefore$ $5 + 6 + 2 = d = 13$

$\therefore$ $a = 1, b = -3, c = -2, d = 13$

Or (P) $\rightarrow$ (3) (Q) $\rightarrow$ (2) (R) $\rightarrow$ (4) (S) $\rightarrow$ (1)

Q. 9. Match List I with List II and select the correct answer using the code given below the lists :

List - I **List - II**

P. Let $y(x)=\cos\left(3\cos^{-1}x\right)$, $x\in[-1,1]$, $x\neq\pm\frac{\sqrt{3}}{2}$. Then **1. 1**

$\frac{1}{y(x)}\left\{\left(x^2-1\right)\frac{d^2y(x)}{dx^2}+x\frac{dy(x)}{dx}\right\}$ equals

Q. Let $A_1, A_2, ..., A_n$ ($n > 2$) be the vertices of a regular **2. 2**

polygon of n sides with its centre at the origin. Let $\vec{a_k}$

be the position vector of the point A_k, k = 1, 2, ..., n.

If $\left|\sum_{k=1}^{n-1}\left(\vec{a_k}\times\vec{a_{k+1}}\right)\right| = \left|\sum_{k=1}^{n-1}\left(\vec{a_k}\cdot\vec{a_{k+1}}\right)\right|$,

then the minimum value of n is

R. If the normal from the point P(h, 1) on the ellipse **3. 8**

$\frac{x^2}{6}+\frac{y^2}{3}=1$ **is perpendicular to the line x + y = 8, then the**

value of h is

S. Number of positive solutions satisfying the equation **4. 9**

$\tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$ is

	P	Q	R	S
(a)	4	3	2	1
(b)	2	4	3	1
(c)	4	3	1	2
(d)	2	4	1	3

Ans. (a)

Solution.

$P(4)\ y=\cos\left(3\cos^{-1}x\right)$

$y=\cos\left[\cos^{-1}\left(4x^3-3x\right)\right]$

$y=4x^3-3x$

$\Rightarrow \dfrac{dy}{dx}=12x^2-3$ and $\dfrac{d^2y}{dx^2}=24x$

$\therefore \dfrac{1}{y}\left\{\left(x^2-1\right)\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}\right\}$

$=\dfrac{1}{4x^3-3x}\left\{\left(x^2-1\right)24x+x\left(12x^2-3\right)\right\}$

$=\dfrac{3x}{4x^3-3x}\left\{8x^2-8+4x^2-1\right\}$

$=\dfrac{3x\left\{12x^2-9\right\}}{4x^3-3x}=\dfrac{9\left\{4x^3-3x\right\}}{4x^3-3x}=9$

Q(3) ... A_1, A_2, A_3, ... A_n are the vertices of a regular $\vec{a}_1, \vec{a}_2$,

.... $\vec{a}_n$ are their position vectors.

$\therefore \quad \left|\vec{a}_1\right|=\left|\vec{a}_2\right|=....=\left|\vec{a}_n\right|=\lambda$

Then $\vec{a}_k\times\vec{a}_{k+1}=\lambda^2\sin\dfrac{2\pi}{n}$

and $\vec{a}_k\times\vec{a}_{k+1}=\lambda^2\cos\dfrac{2\pi}{n}$

Hence given equation reduces to

$(n-1)\lambda^2\sin\dfrac{2\pi}{n}=(n-1)\lambda^2\cos\dfrac{2\pi}{n}$

$\Rightarrow \tan\dfrac{2\pi}{n}=1 \Rightarrow \dfrac{2\pi}{n}=\dfrac{\pi}{4}\Rightarrow n=8$

$R.(2)$ Normal from $P(h, 1)$ on $\frac{x^2}{6}+\frac{y^2}{3}=1$ is

$$\frac{x-h}{h/6}=\frac{y-1}{1/3}$$

$$\Rightarrow 2(x-h)=h(y-1)$$

$$\Rightarrow 2x-hy-h=0$$

It is perpendicular to x + y = 8

$$\therefore \frac{2}{h}\times -1=-1 \Rightarrow h=2$$

$$S.(1)\ \tan^{-1}\left(\frac{1}{2x+1}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{1}{2x+1}+\frac{1}{4x+1}}{1-\frac{1}{2x+1}\cdot\frac{1}{4x+1}}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{4x+1+2x+1}{8x^2+6x+1-1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{6x+2}{8x^2+6x}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \tan^{-1}\left(\frac{3x+1}{4x^2+3x}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$$

$$\Rightarrow \frac{3x+1}{4x^2+3x}=\frac{2}{x^2}$$

$$\Rightarrow 3x^2-7x-6=0 \text{ (for } x>0)$$

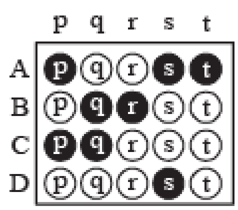
$$\Rightarrow x=3 \text{ or } -\frac{2}{3} \text{ (rejected as } x>0)$$

∴ Only one +ve solution is there

Hence (a) is the correct option.

DIRECTIONS (Q. 10-11) : Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r, s and t. Any given

statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example : If the correct matches are A-p, s and t; B-q and r; C-p and q; and D-s then the correct darkening of bubbles will look like the given.



Q. 10. Match the following :

Column I **Column II**

(A) In R^2, if the magnitude of the projection vector of the vector **(p) 1**

$\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\hat{i}+\hat{j}$ is $\sqrt{3}$ and if $\alpha = 2+\sqrt{3}\beta$, **then possible value of |α| is/are**

(B) Let a and b be real numbers such that the function **(q) 2**

$f(x) = \begin{cases} -3ax^2-2, & x<1 \\ bx+a^2, & x \geq 1 \end{cases}$ if differentiable for all $x \in R$

Then possible value of a is (are)

(C) Let $\omega \neq 1$ be a complex cube root of unity. **(r) 3**

If $(3-3\omega+2\omega^2)^{4n+3}+(2+3\omega-3\omega^2)^{4n+3}+(-3+2\omega+3\omega^2)^{4n+3}=0,$

then possible value (s) of n is (are)

(D) Let the harmonic mean of two positive real numbers a and b be 4. **(s) 4**

If q is a positive real number such that a, 5, q, b is an arithmetic

progression, then the value(s) of | q – a | is (are)

Ans. (A) → q; (B) → p, q; (C) → p, q, s, t; (D) → q, t

Solution.

(A) $\frac{\sqrt{3}\alpha+\beta}{2}=\sqrt{3}\Rightarrow\alpha=\frac{2\sqrt{3}-\beta}{\sqrt{3}}$

$\therefore\quad \frac{2\sqrt{3}-\beta}{\sqrt{3}}=2+\sqrt{3}\beta\Rightarrow\beta=0\Rightarrow\alpha=2$

(B) $Lf'(1)=-6a$ and $Rf'(1)=b$

$-6a=b$ ...(i)

Also f is continuous at $x=1$,

$\therefore -3a-2=b+a^2$

$\Rightarrow a^2-3a+2=0$ (using (i))

$\Rightarrow a = 1, 2$

(C) $(3-3\omega+2\omega^2)^{4n+3}+(2+3\omega-3\omega^2)^{4n+3}+(-3+2\omega+3\omega^2)^{4n+3}=0$

$$\Rightarrow (3-3\omega+2\omega^2)^{4n+3}+\left(\frac{2\omega^2+3-3\omega}{\omega^2}\right)^{4n+3}+\left(\frac{-3\omega+2\omega^2+3}{\omega}\right)^{4n+3}=0$$

$$\Rightarrow (3-3\omega+2\omega^2)^{4n+3}[1+\omega^{4n+3}+(\omega^2)^{4n+3}]=0$$

$\Rightarrow$ 4n + 3 should be an integer other than multiple of 3.

$\therefore$ n = 1, 2, 4, 5

(D) $\frac{2ab}{a+b}=4\Rightarrow ab=2a+2b$ ...(i)

Also $a+q=10$ or $a=10-q$

and $b+5=2q$ or $b=2q-5$

Putting values of a and b in eqn (i)

$q=4$ or $\frac{15}{2}\Rightarrow a=6$ or $\frac{5}{2}$

$\therefore\quad |q-a|=2$ or 5.

Q. 11. Match the following:

Column
I Column II
(A) In a triangle DXYZ, let a, b, and c be the lengths (p) 1

of the sides opposite to the angles X, Y and Z,

respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X-Y)}{\sin Z}$,

, then possible values of n for which $\cos(n\pi\lambda) = 0$ is (are)

(B) In a triangle DXYZ, let a, b and c be the lengths of
the (q) 2

sides opposite to the angles X, Y, and Z respectively. If

1 + cos 2X – 2cos 2Y = 2 sin X sin Y, then possible value

(s) of a/b is (are)

(C) In R^2, let $\sqrt{3}\hat{i}+\hat{j}, \hat{i}+\sqrt{3}\hat{j}$ and $\beta\hat{i}+(1-\beta)\hat{j}$ be the position (r) 3

vectors of X , Y and Z with respect to the origin O,

respectively. If the distance of Z from the bisector of the

acute angle of $\overrightarrow{OX}$ with $\overrightarrow{OY}$ is $\frac{3}{\sqrt{2}}$, then possible

value(s) of |β| is (are)

(D) Suppose that F(a) denotes the area of the region bounded (s)
5

by x = 0, x = 2, $y^2 = 4x$ and $y = |\alpha x - 1| + |\alpha x - 2| + \alpha x$,

Where a∈ {0, 1} . Then the value(s) of $F(\alpha)+\frac{8}{3}\sqrt{2}$, , when

a = 0 and a = 1, is (are)

(t)

6

Ans. (A)→p, r, s; (B) → p; (C) → p, q; (D)→s, t

Solution. (A) $2(a^2 - b^2) = c^2$

$\Rightarrow 2(\sin^2 x - \sin^2 y) = \sin^2 z$

$\Rightarrow 2\sin(x + y)\sin(x - y) = \sin^2 z$

$\Rightarrow 2\sin(x - y) = \sin z \quad (\because \sin(x + y) = \sin z)$

$$\Rightarrow \frac{\sin(x-y)}{\sin z} = \frac{1}{2} = \lambda$$

$$\therefore \cos(n\pi\lambda) = 0 \Rightarrow \cos\frac{n\pi}{2} = 0 \Rightarrow n = 1, 3, 5$$

(B) $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$

$\Rightarrow 2\cos^2 X - 2\cos 2Y = 2\sin X \sin Y$

$\Rightarrow 1 - \sin^2 X - 1 + 2\sin^2 Y = \sin X \sin Y$

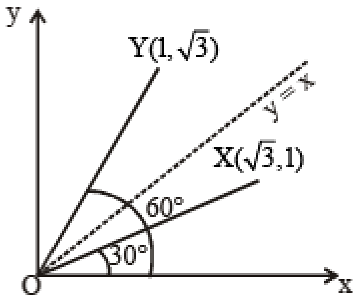
$\Rightarrow \sin^2 X + \sin X \sin Y - 2\sin^2 Y = 0$

$\Rightarrow (\sin X - \sin Y)(\sin X + 2\sin Y) = 0$

$$\Rightarrow \frac{\sin X}{\sin Y} = 1 \text{ or } -2$$

$$\therefore \frac{a}{b} = 1.$$

(C) $X(\sqrt{3}, 1), Y(1, \sqrt{3}), Z(\beta, 1-\beta)$



By symmetry, acute angle bisector of ÐXOY is y = x.

∴ Distance of Z from bisector

$$= \left|\frac{\beta - 1 + \beta}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}} \Rightarrow 2\beta - 1 = \pm 3 \text{ or } \beta = 2 \text{ or } -1$$

∴ $|\beta| = 1, 2$ (D)

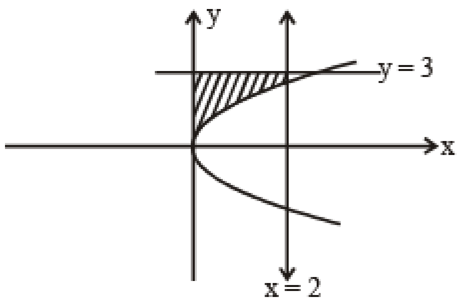
For $\alpha = 0$, $y = 3$

For $\alpha = 1$, $y = |x - 1| + |x - 2| + x$

Case I

F(aα) is the area bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = 3$

$$\therefore \quad F(\alpha) = \int_0^2 (3 - 2\sqrt{x})dx$$



$$= \left|3x - \frac{4x\sqrt{x}}{3}\right|_0^2 = 6 - \frac{8\sqrt{2}}{3}$$

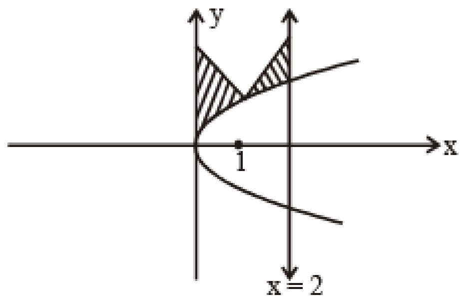
$$\therefore \quad F(\alpha) + \frac{8}{3}\sqrt{2} = 6$$

Case II

F(α) is the area bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = |x - 1| + |x - 2| + x$

$$= \begin{cases} 3 - x, 0 \le x < 1 \\ x + 1, 1 \le x \le 2 \end{cases}$$

$$\therefore \quad F(\alpha) = \int_0^1 (3 - x - 2\sqrt{x})dx + \int_1^2 (x + 1 - 2\sqrt{x})dx$$



$$=\left(3x-\frac{x^2}{2}-\frac{4x}{3}\sqrt{x}\right)_0^1+\left(\frac{x^2}{2}+x-\frac{4}{3}x\sqrt{x}\right)_1^2$$

$$=3-\frac{1}{2}-\frac{4}{3}+2+2-\frac{8\sqrt{2}}{3}-\frac{1}{2}-1+\frac{4}{3}\ =5-\frac{8\sqrt{2}}{3}$$

$$F(\alpha)+\frac{8\sqrt{2}}{3}=5$$

Integer Value Correct Type of Vector Algebra & 3D Geometry

Q. 1. If $\vec{a}$ and $\vec{b}$ are vectors in space given by $\vec{a} = \frac{\hat{i} - 2\hat{j}}{\sqrt{5}}$ **and** $\vec{b} = \frac{2\hat{i} + \hat{j} + 3\hat{k}}{\sqrt{14}}$, **then find the value of** $(2\vec{a} + \vec{b}) \cdot \left[(\vec{a} \times \vec{b}) \times (\vec{a} - 2\vec{b})\right]$. **(2010)**

Ans. 5

Solution.

We have $\vec{a} = \frac{\hat{i} - 2j}{\sqrt{5}}, \vec{b} = \frac{2\hat{i} + j + 3\hat{k}}{\sqrt{14}}$

Clearly $|\vec{a}| = 1$ $|\vec{b}| = 1$ and $\vec{a}.\vec{b} = 0$

$(2\vec{a} + \vec{b}).\left[(\vec{a} \times \vec{b}) \times (\vec{a} - 2\vec{b})\right]$

$= -(2\vec{a} + \vec{b}).\left[(\vec{a} - 2\vec{b}) \times (\vec{a} \times \vec{b})\right]$

$= -(2\vec{a} + \vec{b}).\left[\left((\vec{a} - 2\vec{b}).\vec{b}\right)\vec{a} - \left((\vec{a} - 2\vec{b}).\vec{a}\right)\vec{b}\right]$

$= -(2\vec{a} + \vec{b}).\left[\left(\vec{a}.\vec{b} - 2|\vec{b}|^2\right)\vec{a} - \left(|\vec{a}|^2 - 2\vec{b}.\vec{a}\right)\vec{b}\right]$

$= -(2\vec{a} + \vec{b}).\left[-2\vec{a} - \vec{b}\right]$

$= (2\vec{a} + \vec{b}).(2\vec{a} + \vec{b}) = 4|\vec{a}|^2 + |\vec{b}|^2 \quad (\because \vec{a}.\vec{b} = 0)$

$= 4 + 1 = 5.$

Q. 2. If the distance between the plane Ax – 2y + z = d and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ **and** $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$**, then find |d|. (2010)**

Ans. 6

Solution. The equation of plane containing the lines

$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}=0 \Rightarrow x-2y+z=0$$

∴ Distance between x – 2 y +z=0 and Ax –2 y + z = d

= Perpendicular distance between parallel planes (∴ A= 1)

$$=\frac{|d|}{\sqrt{6}}=\sqrt{6} \Rightarrow |d|=6.$$

Q. 3. Let $\vec{a}=-\vec{i}-\hat{k}, \vec{b}=-\hat{i}+\hat{j}$ and $\vec{c}=\hat{i}+2\hat{j}+3\hat{k}$ **be three given vectors.** If $\vec{r}$ **is a vector such that** $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$ and $\vec{r}.\vec{a}=0$, **then the value of** $\vec{r}.\vec{b}$ is **(2011)**

Ans. 9

Solution.

We have $\vec{r}\times\vec{b}-\vec{c}\times\vec{b}=\vec{0}$

$\Rightarrow (\vec{r}-\vec{c})\times\vec{b}=\vec{0} \Rightarrow \vec{r}-\vec{c} \parallel \vec{b}$

Let $\vec{r}-\vec{c}=\lambda\vec{b}$ or $\vec{r}=\vec{c}+\lambda\vec{b}$

$\Rightarrow \vec{r}=\hat{i}+2\hat{j}+3\hat{k}-\lambda\hat{i}+\lambda\hat{j}=(1-\lambda)i+(2+\lambda)\hat{j}+3\hat{k}$

$\vec{r}.\vec{a}=0 \Rightarrow -1+\lambda-3=0 \Rightarrow \lambda=4$

$\therefore \vec{r}=-3\hat{i}+6\hat{j}+3\hat{k}$

$\therefore \vec{r}.\vec{b}=3+6=9$

Q. 4. If $\vec{a}, \vec{b}$ and $\vec{c}$ **are unit vectors satisfying** $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2=9$, then $|2\vec{a}+5\vec{b}+5\vec{c}|$ is **(2012)**

Ans. 3

Solution. $\vec{a}, \vec{b}, \vec{c}$ are units vectors such that

$$|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2=9$$

$$\Rightarrow 2\left(|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2\right)-2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})=9$$

$$\Rightarrow \vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}=\frac{-3}{2}$$

Also $|\vec{a}+\vec{b}+\vec{c}|^2=|\vec{a}|^2+|\vec{b}|^2+|\vec{c}|^2+2(\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a})$

$$=1+1+1+2\times\left(-\frac{3}{2}\right)=0$$

$$\Rightarrow \vec{a}+\vec{b}+\vec{c}=0 \Rightarrow (\vec{b}+\vec{c})=-\vec{a}$$

$$\therefore \left|2\vec{a}+5(\vec{b}+\vec{c})\right|=|2\vec{a}-5\vec{a}|=|-3\vec{a}|=3$$

Q. 5. Consider the set of eight vectors $V=\left\{a\hat{i}+b\hat{j}+c\hat{k}:a,b,c\in\{-1,1\}\right\}$. **Three non- coplanar vectors can be chosen from V in 2^p ways. Then p is (JEE Adv. 2013)**

Ans. 5

Solution. Given 8 vectors are

(1, 1, 1), (–1, –1, –1); (–1, 1, 1), (1, –1, –1); (1, –1, 1),

(–1, 1, –1); (1, 1, –1), (–1, –1, 1)

These are 4 diagonals of a cube and their opposites.

For 3 non coplanar vectors first we select 3 groups of diagonals and its opposite in $^4C_3$ ways. Then one vector from each group can be selected in 2 × 2 × 2 ways.

∴ Total ways = $^4C_3 \times 2 \times 2 \times 2 = 32 = 25$

∴ p = 5

Q. 6. A pack contains n cards numbered from 1 to n. Two consecutive numbered cards are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is k, then k – 20 = (JEE Adv. 2013)

Ans. 5

Solution. Let k, k + 1 be removed from pack.

∴ (1 + 2 + 3 + ... + n) – (k + k + 1) = 1224

$$\frac{n(n+1)}{2} - 2k = 1225 \Rightarrow k = \frac{n(n+1) - 2450}{4}$$

$$\text{for } n = 50,\ k = 25 \ \therefore\ k - 20 = 5$$

Q. 7. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be three non-coplanar unit vectors such that the angle between every pair of them is π/3. If $\vec{a}\times\vec{b}+\vec{b}\times\vec{c} = p\vec{a}+q\vec{b}+r\vec{c}$, where p, q and r are scalars, then the value of $\frac{p^2+2q^2+r^2}{q^2}$ is (JEE Adv. 2014)

Ans. 4

Solution.

$$\vec{a}.\vec{b} = \vec{b}.\vec{c} = \vec{c}.\vec{a} = \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\text{Given } p\vec{a}+q\vec{b}+r\vec{c} = \vec{a}\times\vec{b}+\vec{b}\times\vec{c}$$

Taking its dot product with $\vec{a}, \vec{b}, \vec{c}$, we get

$$p+\frac{1}{2}q+\frac{1}{2}r = \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] \quad ...(1)$$

$$\frac{1}{2}p+q+\frac{1}{2}r = 0 \quad ...(2)$$

$$\frac{1}{2}p+\frac{1}{2}q+r = \left[\vec{a}\ \ \vec{b}\ \ \vec{c}\right] \quad ...(3)$$

From (1) and (3), p = r Using (2) q = – p

$$\therefore \frac{p^2+2q^2+r^2}{q^2} = \frac{p^2+2p^2+p^2}{p^2} = 4$$

Q. 8. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ **are three non-coplanar vectors in** $\mathbb{R}^3$. **Let the components of a vector** $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ **be 4, 3 and 5, respectively. If the components of this vector** $\vec{s}$ along $(-\vec{p}+\vec{q}+\vec{r}), (\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}-\vec{q}+\vec{r})$ **are x, y and z, respectively, then the value of 2x + y + z is (JEE Adv. 2015)**

Ans. 9

Solution. $\vec{s} = 4\vec{p} + 3\vec{q} + 5\vec{r}$

$$\vec{s} = x(-\vec{p}+\vec{q}+\vec{r}) + y(\vec{p}-\vec{q}+\vec{r}) + z(-\vec{p}-\vec{q}+\vec{r})$$

$\Rightarrow -x + y - z = 4$

$x - y - z = 3$

$x + y + z = 5$

Solving above equation $x = 4, y = \frac{9}{2}, z = \frac{-7}{2}$

$\therefore 2x + y + z = 9$